

ON TENSOR PRODUCTS OF A MINIMAL AFFINIZATION WITH AN EXTREME KIRILLOV-RESHETIKHIN MODULE FOR TYPE A

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ABSTRACT. For a quantum affine algebra of type A , we describe the composition series of the tensor product of a general minimal affinization with a Kirillov-Reshetikhin module associated to an extreme node of the Dynkin diagram of the underlying simple Lie algebra.

1. INTRODUCTION

Unraveling the intricate structure of the category of finite-dimensional representations of quantum affine algebras has drawn the attention of experts in representation theoretic Lie theory since the early 1990s. One of the interesting problems to be addressed is that of understanding the class of irreducible affinizations of a given simple module for the quantum group $U_q(\mathfrak{g})$ associated to the underlying finite-dimensional simple Lie algebra \mathfrak{g} . In particular, one wants to classify and describe the structure of the minimal such affinizations in the sense defined by Chari in [3]. The early work of Chari and Pressley in this direction [9, 10] describes the classification of the Drinfeld polynomials corresponding to such minimal affinizations in the case that \mathfrak{g} is not of types D or E . It turns out that, in those cases, there exists essentially one minimal affinization for any given simple module of $U_q(\mathfrak{g})$, i.e., all minimal affinizations of that module are isomorphic as modules for $U_q(\mathfrak{g})$ (affinizations which are isomorphic as modules for $U_q(\mathfrak{g})$ are said to be equivalent). These papers also classify the minimal affinizations of the so called regular representations in types D and E . Namely, those for which the support of corresponding highest weight either does not bound a subdiagram of type D or, if it does, then the trivalent node belongs to the support. The number of equivalence classes of minimal affinizations in types D and E depends on the highest weight. If its support does not bound a subdiagram of type D , there is only one class as before, but if it does, then there are typically 3 equivalent classes (essentially coming from the symmetry of the subdiagram of type D_4 , even if the support is not symmetric).

Thus, as long as classification goes, it remains to study the irregular minimal affinizations in types D and E . This paper is the first part of our work towards the classification of minimal affinizations in type D . It contains results in the case that \mathfrak{g} is of type A which are crucial to obtain such classification. Namely, by looking at the 3 connected components of the diagram of type D after removing the trivalent node, we have a diagram of type A_n , $n \geq 1$, and two of type A_1 . The minimal affinizations are realizable as a simple factor of the tensor product of simple modules supported in each of these connected components. The strategy is to compare “all” possible such tensor products to pinpoint which ones give rise to minimal affinizations. The tensor product of only two of the factors can be regarded as a module for a diagram-subalgebra of type A . This partially explains our interest in tensor products of a general minimal affinization with a Kirillov-Reshetikhin module

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associated to an extreme node of the Dynkin diagram for type A (a Kirillov-Reshetikhin module is a minimal affinization of a simple module whose highest weight is a multiple of a fundamental weight). Since these results are interesting in their own right and have strong potential to be useful for studying other aspects of the category of finite-dimensional representations of quantum affine algebras (in all types) and the proofs are quite long, we present them here by themselves. The classification of minimal affinizations for type D will appear in a forthcoming publication (see also [28]).

It is important to remark that tensor products of irreducible representations of quantum affine algebras in general, and of minimal affinizations in particular, is a relevant topic not only to the understanding of the underlying category of finite-dimensional representations, but it also has very important applications or deep connections to other areas such as integrable systems in mathematical systems, combinatorics, and cluster algebras. The most studied case is that of tensor products of Kirillov-Reshetikhin modules. Such tensor products give rise to a remarkable family of short exact sequences which can be encoded in a set of recurrence relations, called T -systems, which have many applications in integrable system. The literature in this direction is vast and we refer the reader to [15, 23] for more details and references. On the other hand, the connection to cluster algebras was first discovered in [18]. More recently, it has been shown in [19] that the T -systems can be interpreted as cluster transformations in a cluster algebra having KR-modules correspond to an initial seed. This then lead to an algorithm for computing the q -characters of KR-modules by successive approximations via the combinatorics of cluster algebras. It would then be interesting to eventually study the results of the present paper from the perspective of T -systems and cluster algebras. The connection of graded limits of tensor products of finite-dimensional representations of quantum affine algebras with the notion of fusion products in the sense of [12] is also another topic of recent interest (see for instance [1, 2, 26, 27] and references therein). Thus, it should also be interesting to study the graded limits of the tensor products studied here in that context as well.

We now describe the organization and the main results of this paper. In Section 2, we fix the notation related to the study of finite-dimensional representations of quantum affine algebras, review the basic facts about such representations as well as the relevant known facts about minimal affinizations, and state the main result of the paper (the combination of Theorem 2.5.1 with Corollary 2.5.2). The statement can be informally described as follows. The tensor product of an “increasing” minimal affinization with a KR-module supported at the last node of the Dynkin diagram is an indecomposable module of length at most 2. We describe precisely the conditions on the Drinfeld polynomials of the tensor factors which give rise to a length-2 tensor product and write down an explicit formula for the Drinfeld polynomial of the “extra” irreducible factor. Moreover, we precisely describe the socle and the head for both orders of the tensor factors. The other possibilities of tensor products (replacing the minimal affinization by a “decreasing” one or the KR-module by one supported at the first node) can be obtained from the case established in Theorem 2.5.1 by certain duality arguments and the precise explanation and statements are given in Sections 5.1 and 5.2. In Section 3, we review results about the main tool we shall use in the proof of Theorem 2.5.1: the theory of q -characters. In particular, and very importantly, in Section 3.3, we review the description of the q -character of a minimal affinization in type A in terms of semi-standard tableaux. The core of our proof is based on combinatorial analysis of “products” of such tableaux. The proof of Theorem 2.5.1 is given in Section 4. After explaining the general scheme of the proof in Section 4.1, we proceed by describing the dominant ℓ -weights of the tensor product in Sections 4.2 and 4.3. It turns out that the set of such dominant ℓ -weights is totally ordered and the corresponding ℓ -weight spaces are one-dimensional (this is the statement of Proposition 4.1.1, which is also an interesting result by itself and can be considered the second most important result of the paper). In the last step of the proof, performed in Section 4.4, we start by explicitly describing which of these dominant ℓ -weights are ℓ -weights of the obvious irreducible factor of the tensor product (the one whose Drinfeld polynomial is the product of those of the two tensor factors). Under certain

conditions on the Drinfeld polynomials of the tensor factors (the conditions in the statement of Theorem 2.5.1), we see that not all the dominant ℓ -weights are ℓ -weights of the obvious irreducible factor. Hence, the highest of the remaining ones must be the Drinfeld polynomial of an extra irreducible factor and we show that all remaining dominant ℓ -weights are ℓ -weights of this extra irreducible factor. The proof of Corollary 2.5.2, about the dependence of the socle and the head on the order of the tensor factors, is given in Section 5.3.

In principle, the methods we employed here could be used to obtain similar information about the tensor product of any two minimal affinizations. However, the combinatorics would be substantially more complicated and it is unclear if it would be manageable to obtain results as precise as we did. Also, it is unlikely that Proposition 4.1.1 remains valid and multiplicity issues could turn the arguments we employed here insufficient. In light of our remarks about T -system and cluster algebras above, our main theorem may be regarded as a step towards studying short exact sequences related to tensor products of minimal affinizations beyond KR-modules (see also [23]) and, hopefully, the machinery of cluster algebras may eventually provide more tools to expand the scope of the study initiated here.

2. BASIC NOTATION AND THE MAIN THEOREM

Throughout the paper, let $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_{\geq m}$ denote the sets of complex numbers, reals, integers, and integers bigger or equal m , respectively. Given a ring \mathbb{A} , the underlying multiplicative group of units is denoted by \mathbb{A}^\times . The dual of a vector space V is denoted by V^* . The symbol \cong means “isomorphic to”.

2.1. The Algebras. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, $I = \{1, \dots, n\}$, and \mathfrak{h} the standard Cartan subalgebra, i.e., \mathfrak{h} is the span of h_1, \dots, h_n with $h_i = e_{i,i} - e_{i+1,i+1}$, $i \in I$, where $e_{i,j}$ is the matrix whose (i, j) entry is δ_{ij} . Fix the set of positive roots R^+ so that positive root vectors are upper triangular matrices and let

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha} \quad \text{where} \quad \mathfrak{g}_{\pm\alpha} = \{x \in \mathfrak{g} : [h, x] = \pm\alpha(h)x, \forall h \in \mathfrak{h}\}.$$

The simple roots will be denoted by α_i and the fundamental weights by ω_i , $i \in I$. Q, P, Q^+, P^+ will denote the root and weight lattices with corresponding positive cones, respectively. Equip \mathfrak{h}^* with the partial order $\lambda \leq \mu$ iff $\mu - \lambda \in Q^+$. Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of \mathfrak{g} , i.e., $c_{ij} = \alpha_j(h_i)$. The Weyl group is denoted by \mathcal{W} .

If \mathfrak{a} is a Lie algebra over \mathbb{C} , define its loop algebra to be $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ with bracket given by $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$. Clearly $\mathfrak{a} \otimes 1$ is a subalgebra of $\tilde{\mathfrak{a}}$ isomorphic to \mathfrak{a} and, by abuse of notation, we will continue denoting its elements by x instead of $x \otimes 1$. Then $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra.

Fix $q \in \mathbb{C}^\times$ which is not a root of unity and set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]! = [m][m-1] \dots [2][1], \quad \begin{bmatrix} m \\ r \end{bmatrix} = \frac{[m]!}{[r]![m-r]!},$$

for $r, m \in \mathbb{Z}_{\geq 0}$, $m \geq r$.

The quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ is the associative \mathbb{C} -algebra with generators $x_{i,r}^\pm$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z} \setminus \{0\}$) and the following defining relations:

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i h_{j,r} = h_{j,r} k_i, \quad [h_{i,r}, h_{j,s}] = 0, \\ k_i x_{j,r}^\pm k_i^{-1} &= q^{\pm c_{ij}} x_{j,r}^\pm, \quad [h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [r c_{ij}] x_{j,r+s}^\pm, \end{aligned}$$

$$\begin{aligned}
x_{i,r+1}^{\pm} x_{j,s}^{\pm} - q^{\pm c_{ij}} x_{j,s}^{\pm} x_{i,r+1}^{\pm} &= q^{\pm c_{ij}} x_{i,r}^{\pm} x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm} x_{i,r}^{\pm}, \\
[x_{i,r}^{\pm}, x_{j,s}^{\pm}] &= 0, \quad \text{if } c_{ij} = 0, \quad \text{whlie, if } c_{ij} = -1, \\
x_{i,r_1}^{\pm} x_{i,r_2}^{\pm} x_{j,s}^{\pm} + x_{i,r_2}^{\pm} x_{i,r_1}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r_1}^{\pm} x_{i,r_2}^{\pm} + x_{j,s}^{\pm} x_{i,r_2}^{\pm} x_{i,r_1}^{\pm} &= [2] \left(x_{i,r_1}^{\pm} x_{j,s}^{\pm} x_{i,r_2}^{\pm} + x_{i,r_2}^{\pm} x_{j,s}^{\pm} x_{i,r_1}^{\pm} \right), \\
[x_{i,r}^+, x_{j,s}^-] &= \delta_{i,j} \frac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q - q^{-1}},
\end{aligned}$$

where the $\psi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\Psi_i^{\pm}(u) = \sum_{r \in \mathbb{Z}} \psi_{i,\pm r}^{\pm} u^r = k_i^{\pm 1} \exp \left(\pm (q - q^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^s \right).$$

In particular, $\psi_{i,\pm r}^{\pm} = 0$ if $r < 0$.

Denote by $U_q(\tilde{\mathfrak{n}}^{\pm}), U_q(\tilde{\mathfrak{h}})$ the subalgebras of $U_q(\tilde{\mathfrak{g}})$ generated by $\{x_{i,r}^{\pm}\}, \{k_i^{\pm 1}, h_{i,s}\}$, respectively. Let $U_q(\mathfrak{g})$ be the subalgebra generated by $x_i^{\pm} := x_{i,0}^{\pm}, k_i^{\pm 1}, i \in I$, and define $U_q(\mathfrak{n}^{\pm}), U_q(\mathfrak{h})$ in the obvious way. $U_q(\mathfrak{g})$ is a subalgebra of $U_q(\tilde{\mathfrak{g}})$ and multiplication establishes isomorphisms of vector spaces:

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \quad \text{and} \quad U_q(\tilde{\mathfrak{g}}) \cong U_q(\tilde{\mathfrak{n}}^-) \otimes U_q(\tilde{\mathfrak{h}}) \otimes U_q(\tilde{\mathfrak{n}}^+).$$

For $i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$, define $(x_{i,r}^{\pm})^{(k)} = \frac{(x_{i,r}^{\pm})^k}{[k]!}$. Define also elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$ by equating powers of u in the formal power series

$$(2.1.1) \quad \Lambda_i^{\pm}(u) = \sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp \left(- \sum_{s=1}^{\infty} \frac{h_{i,\pm s}}{[s]} u^s \right).$$

The elements $\Lambda_{i,\pm r}$ together with $k_i^{\pm 1}, i \in I, r \in \mathbb{Z}$, generate $U_q(\tilde{\mathfrak{h}})$ as an algebra.

The following assignments,

$$\begin{aligned}
\Delta(x_i^+) &= x_i^+ \otimes 1 + k_i \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
S(x_i^+) &= -k_i^{-1} x_i^+, \quad S(x_i^-) = -x_i^- k_i, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}, \\
\varepsilon(x_i^{\pm}) &= 0, \quad \varepsilon(k_i^{\pm 1}) = 1,
\end{aligned}$$

for all $i \in I$, define a structure of Hopf algebra in $U_q(\mathfrak{g})$, where Δ is the co-multiplication, ε is the co-unity and S is the antipode. The algebra $U_q(\tilde{\mathfrak{g}})$ is also a Hopf algebra and the structure maps can be described exactly as above using the Chevalley-Kac generators. However, a precise expression for the comultiplication in terms of the generators $x_{i,r}^{\pm}, h_{i,r}, k_i^{\pm 1}$ is not known (see [8] and references therein). $U_q(\mathfrak{g})$ is a Hopf subalgebra of $U_q(\tilde{\mathfrak{g}})$.

2.2. The ℓ -Weight Lattice. Consider the multiplicative group \mathcal{P} of n -tuples of rational functions $\boldsymbol{\mu} = (\mu_1(u), \dots, \mu_n(u))$ with values in \mathbb{C} such that $\mu_i(0) = 1$ for all $i \in I$. We shall refer to \mathcal{P} as the ℓ -weight lattice of $U_q(\tilde{\mathfrak{g}})$, to the elements of \mathcal{P} as ℓ -weight and to elements of the submonoid \mathcal{P}^+ of \mathcal{P} consisting of n -tuples of polynomials as dominant ℓ -weights. Given $a \in \mathbb{C}^{\times}$ and $i \in I$, define the fundamental ℓ -weight $\omega_{i,a} \in \mathcal{P}$ by

$$(\omega_{i,a})_j(u) = 1 - \delta_{i,j} a u.$$

Clearly, \mathcal{P} is the free abelian group generated by these elements. If

$$(2.2.1) \quad \boldsymbol{\mu} = \prod_{(i,a) \in I \times \mathbb{C}^{\times}} \omega_{i,a}^{p_{i,a}}$$

we shall say that $\omega_{i,a}$ (respectively, $\omega_{i,a}^{-1}$) appears in $\boldsymbol{\mu}$ if $p_{i,a} > 0$ (respectively, $p_{i,a} < 0$).

Consider the group homomorphism (weight map) $\text{wt} : \mathcal{P} \rightarrow P$ by setting $\text{wt}(\omega_{i,a}) = \omega_i$. Given $\omega \in \mathcal{P}^+$ with $\omega_i(u) = \prod_j (1 - a_{i,j}u)$, where $a_{i,j} \in \mathbb{C}$, let $\omega^- \in \mathcal{P}^+$ be defined by $\omega_i^-(u) = \prod_j (1 - a_{i,j}^{-1}u)$. For convenience, we will sometimes use the notation $\omega^+ = \omega$. Given $\mu \in \mathcal{P}$, say $\mu = \omega\varpi^{-1}$ with $\omega, \varpi \in \mathcal{P}^+$, define a \mathbb{C} -algebra homomorphism $\Psi_\mu : U_q(\tilde{\mathfrak{h}}) \rightarrow \mathbb{C}$ by setting $\Psi_\mu(k_i^{\pm 1}) = q^{\pm \text{wt}(\mu)(h_i)}$ and

$$(2.2.2) \quad \Psi_\mu(\Lambda_i^\pm(u)) = \frac{(\omega^\pm)_i(u)}{(\varpi^\pm)_i(u)}.$$

One easily checks that the map $\Psi : \mathcal{P} \rightarrow (U_q(\tilde{\mathfrak{h}}))^*$ given by $\mu \mapsto \Psi_\mu$ is injective. From now on we will identify \mathcal{P} with its image in $(U_q(\tilde{\mathfrak{h}}))^*$ under Ψ .

It will be convenient to introduce the following notation. Given $i \in I, a \in \mathbb{C}^\times, m \in \mathbb{Z}_{\geq 0}$, define

$$\omega_{i,a,m} = \prod_{j=0}^{m-1} \omega_{i,aq^{m-1-2j}}.$$

Also, following [5], define

$$\alpha_{i,a} = \omega_{i,aq,2} \prod_{j \neq i} \omega_{j,aq,-c_{j,i}}^{-1} = \omega_{i,a} \omega_{i,aq^2} \prod_{j:|j-i|=1} \omega_{j,aq}^{-1}.$$

We shall refer to $\alpha_{i,a}$ as a simple ℓ -root. The subgroup of \mathcal{P} generated by the simple ℓ -roots is called the ℓ -root lattice of $U_q(\tilde{\mathfrak{g}})$ and will be denoted by \mathcal{Q} . Let also \mathcal{Q}^+ be the submonoid generated by the simple ℓ -roots. Quite clearly $\text{wt}(\alpha_{i,a}) = \alpha_i$. Define a partial order on \mathcal{P} by

$$\mu \leq \omega \quad \text{if} \quad \omega\mu^{-1} \in \mathcal{Q}^+.$$

It is well-known that the elements $\alpha_{i,a}$ are multiplicatively independent, i.e., if $(i_j, a_j), j = 1, \dots, m$, is a family of distinct elements of $I \times \mathbb{C}^\times$, then

$$(2.2.3) \quad \prod_{j=1}^m \alpha_{i_j, a_j}^{k_j} = 1 \Leftrightarrow k_j = 0 \text{ for all } j = 1, \dots, m.$$

For further convenience of notation, given $1 \leq i \leq j \leq n, a \in \mathbb{C}^\times$, we set

$$(2.2.4) \quad \alpha_{i,j,a} = \prod_{k=i}^j \alpha_{k,aq^{k-i}} \quad \text{and} \quad \alpha_{j,i,a} = \prod_{k=i}^j \alpha_{j+i-k,aq^{k-i}}.$$

2.3. Finite-Dimensional Representations. We start by reviewing some basic facts about finite-dimensional representations of $U_q(\mathfrak{g})$. For the details see [8] for instance.

Given a $U_q(\mathfrak{g})$ -module V and $\mu \in P$, let

$$V_\mu = \{v \in V : k_i v = q^{\mu(h_i)} v \text{ for all } i \in I\}.$$

A nonzero vector $v \in V_\mu$ is called a weight vector of weight μ . If v is a weight vector such that $x_i^+ v = 0$ for all $i \in I$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ , then V is said to be a highest-weight module of highest weight λ . A $U_q(\mathfrak{g})$ -module V is said to be a weight module if $V = \bigoplus_{\mu \in P} V_\mu$. Denote by \mathcal{C}_q be the category of all finite-dimensional weight modules of $U_q(\mathfrak{g})$. The following theorem summarizes the basic facts about \mathcal{C}_q .

Theorem 2.3.1. Let V be an object of \mathcal{C}_q . Then:

- (a) $\dim V_\mu = \dim V_{w\mu}$ for all $w \in \mathcal{W}$.
- (b) V is completely reducible.

(c) For each $\lambda \in P^+$ the $U_q(\mathfrak{g})$ -module $V_q(\lambda)$ generated by a vector v satisfying

$$x_i^+ v = 0, \quad k_i v = q^{\lambda(h_i)} v, \quad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \forall i \in I,$$

is nonzero, irreducible, and finite-dimensional. If $V \in \mathcal{C}_q$ is irreducible, then V is isomorphic to $V_q(\lambda)$ for a unique $\lambda \in P^+$. \diamond

We now turn to finite-dimensional representations of $U_q(\tilde{\mathfrak{g}})$. Let V be a $U_q(\tilde{\mathfrak{g}})$ -module. We say that a nonzero vector $v \in V$ is an ℓ -weight vector if there exists $\omega \in \mathcal{P}$ and $k \in \mathbb{Z}_{>0}$ such that

$$(\eta - \Psi_\omega(\eta))^k v = 0 \quad \text{for all } \eta \in U_q(\tilde{\mathfrak{h}}).$$

In that case, ω is said to be the ℓ -weight of v . V is said to be an ℓ -weight module if every vector of V is a linear combination of ℓ -weight vectors. In that case, let V_ω denote the subspace spanned by all ℓ -weight vectors of ℓ -weight ω . An ℓ -weight vector v is said to be a highest- ℓ -weight vector if

$$\eta v = \Psi_\omega(\eta) v \quad \text{for every } \eta \in U_q(\tilde{\mathfrak{h}}) \quad \text{and} \quad x_{i,r}^+ v = 0 \quad \text{for all } i \in I, r \in \mathbb{Z}.$$

V is said to be a highest- ℓ -weight module if it is generated by a highest- ℓ -weight vector. Denote by $\tilde{\mathcal{C}}_q$ the category of all finite-dimensional ℓ -weight modules of $U_q(\tilde{\mathfrak{g}})$. $\tilde{\mathcal{C}}_q$ is an abelian category stable under tensor product [14].

Observe that if $V \in \tilde{\mathcal{C}}_q$, then $V \in \mathcal{C}_q$ and

$$(2.3.1) \quad V_\lambda = \bigoplus_{\omega: \text{wt}(\omega)=\lambda} V_\omega.$$

Moreover, if V is a highest- ℓ -weight module of highest ℓ -weight ω , then

$$(2.3.2) \quad \dim(V_{\text{wt}(\omega)}) = 1 \quad \text{and} \quad V_\mu \neq 0 \Rightarrow \mu \leq \text{wt}(\omega).$$

The next proposition is easily established using (2.3.2).

Proposition 2.3.2. If V is a highest- ℓ -weight module, then it has a unique maximal proper submodule and, hence, a unique irreducible quotient. \diamond

Given $\omega \in \mathcal{P}^+$, the Weyl module $W_q(\omega)$ is the $U_q(\tilde{\mathfrak{g}})$ -module defined by the quotient of $U_q(\tilde{\mathfrak{g}})$ by the left ideal generated by the elements

$$x_{i,r}^+, \quad (x_i^-)^{\text{wt}(\omega)(h_i)+1}, \quad \eta - \Psi_\omega(\eta), \quad i \in I, r \in \mathbb{Z}, \eta \in U_q(\tilde{\mathfrak{h}}).$$

In particular, it is a highest- ℓ -weight module. Denote by $V_q(\omega)$ the irreducible quotient of $W_q(\omega)$. The next theorem was proved in [11] and recovers the classification of the simple objects of $\tilde{\mathcal{C}}_q$ obtained previously in [6].

Theorem 2.3.3. (a) For every $\omega \in \mathcal{P}^+$, the module $W_q(\omega)$ is nonzero and, moreover, it is the universal finite-dimensional $U_q(\tilde{\mathfrak{g}})$ -module with highest ℓ -weight ω .

(b) If V is a simple object of $\tilde{\mathcal{C}}_q$, there exists unique $\omega \in \mathcal{P}^+$ such that $V \cong V_q(\omega)$. \diamond

2.4. Minimal Affinizations. We now review the notion of minimal affinizations of an irreducible $U_q(\mathfrak{g})$ -module introduced in [3].

Given $\lambda \in P^+$, a $U_q(\tilde{\mathfrak{g}})$ -module V is said to be an affinization of $V_q(\lambda)$ if there exists an isomorphism of $U_q(\mathfrak{g})$ -module,

$$(2.4.1) \quad V \cong V_q(\lambda) \oplus \bigoplus_{\mu < \lambda} V_q(\mu)^{\oplus m_\mu(V)}$$

for some $m_\mu(V) \in \mathbb{Z}_{\geq 0}$. Two affinizations of $V_q(\lambda)$ are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$ -modules. Notice that a highest- ℓ -weight module of highest ℓ -weight $\omega \in \mathcal{P}^+$ is an affinization of $V_q(\lambda)$ if and only if $\text{wt}(\omega) = \lambda$.

The partial order on P^+ induces a natural partial order on the set of (equivalence classes of) affinizations of $V_q(\lambda)$. Namely, if V and W are affinizations of $V_q(\lambda)$, say that $V \leq W$ if one of the following conditions hold:

- (i) $m_\mu(V) \leq m_\mu(W)$ for all $\mu \in P^+$;
- (ii) for all $\mu \in P^+$ such that $m_\mu(V) > m_\mu(W)$ there exists $\nu > \mu$ such that $m_\nu(V) < m_\nu(W)$.

A minimal element of this partial order is said to be a minimal affinization. Clearly, a minimal affinization of $V_q(\lambda)$ must be irreducible as a $U_q(\tilde{\mathfrak{g}})$ -module and, hence, it is of the form $V_q(\omega)$ for some $\omega \in \mathcal{P}^+$ such that $\text{wt}(\omega) = \lambda$.

Given $i, j \in I, i \leq j$, and $\lambda \in P$, set

$${}_i|\lambda|_j = \sum_{k=i}^j \lambda(h_k).$$

If $i = 1$, we simplify notation and write $|\lambda|_j$ instead of ${}_1|\lambda|_j$ and similarly if $j = n$. For $i > j$ we set ${}_i|\lambda|_j = 0$. Set also

$$p_{i,j}(\lambda) = {}_{i+1}|\lambda|_j + {}_i|\lambda|_{j-1} + j - i \quad \text{if } i \leq j$$

and $p_{i,j}(\lambda) = p_{j,i}(\lambda)$ if $j < i$. Notice that, if $i = j$, then $p_{i,j}(\lambda) = 0$ and

$$p_{i,j}(\lambda) = \lambda(h_i) + \lambda(h_j) + 2 {}_{i+1}|\lambda|_{j-1} + j - i \quad \text{if } i < j.$$

The next theorem proved in [7, Theorem 2.9] (see also [9]) gives the classification of the minimal affinizations in type A .

Theorem 2.4.1. For every $\lambda \in P^+$, there exists a unique class of minimal affinizations of $V_q(\lambda)$. Moreover, $V_q(\omega)$ is a minimal affinization of $V_q(\lambda)$ if and only if there exist $a_i \in \mathbb{C}^\times, i \in I$, and $\epsilon = \pm 1$ such that

$$(2.4.2) \quad \omega = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)} \quad \text{with} \quad \frac{a_i}{a_j} = q^{\epsilon p_{i,j}(\lambda)} \quad \text{for all } i < j.$$

In that case, $V_q(\omega) \cong V_q(\lambda)$ as representations of $U_q(\mathfrak{g})$. \diamond

Notice that (2.4.2) is equivalent to saying that there exist $a \in \mathbb{C}^\times$ and $\epsilon = \pm 1$ such that

$$(2.4.3) \quad \omega = \prod_{i \in I} \omega_{i, a_i, \lambda(h_i)} \quad \text{with} \quad a_i = a q^{\epsilon p_{i,n}(\lambda)} \quad \text{for all } i \in I.$$

The support of $\mu \in P$ is defined by

$$\text{supp}(\mu) = \{i \in I : \mu(h_i) \neq 0\}.$$

Note that if $\# \text{supp}(\lambda) > 1$, the pair (a, ϵ) in (2.4.3) is unique. In that case, if ω satisfies (2.4.2) with $\epsilon = 1$, we say that $V_q(\omega)$ is a decreasing minimal affinization (because the powers of q in (2.4.3) decrease as i increases). Otherwise, we say $V_q(\omega)$ is an increasing minimal affinization. However, if $\# \text{supp}(\lambda) = 1$, ω can be represented in the form (2.4.3) by two choices of pairs (a, ϵ) , one for each value of ϵ . We do not fix a preferred presentation in that case. The minimal affinizations satisfying $\# \text{supp}(\lambda) \leq 1$ are called Kirillov-Reshetikhin modules.

2.5. The Main Theorem. Fix ω as in (2.4.2) with $\epsilon = -1$ and set

$$i_0 = \max(\text{supp}(\lambda)), \quad a = a_{i_0}.$$

Thus,

$$(2.5.1) \quad \omega = \prod_{i \in \text{supp}(\lambda)} \omega_{i, a q^{s_i}, \lambda(h_i)} \quad \text{with} \quad s_i = -p_{i, i_0}(\lambda).$$

For notational convenience, we define s_i by (2.5.1) for all $1 \leq i \leq i_0$. Fix also

$$\varpi = \omega_{n,b,k} \quad \text{for some} \quad b \in \mathbb{C}^\times, k \in \mathbb{Z}_{>0},$$

and set

$$\lambda = \omega \varpi \quad \text{and} \quad V = V_q(\omega) \otimes V_q(\varpi).$$

Theorem 2.5.1. V is reducible if and only if there exist $s \in \mathbb{Z}, p \in \text{supp}(\lambda)$, and $k' > 0$ such that $b = aq^s$ and either one of the following options hold:

- (i) $k' \leq \min\{\lambda(h_p), k\}$ and $s + k + n - p + 2 = s_p - \lambda(h_p) + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$, $\lambda(h_{i_0}) + n - i_0 + 2 = s - k + 2k'$, and $p = \max\{i \in I : |\lambda| \geq k'\}$.

In both cases, V has length 2 and the highest ℓ -weight λ' of the simple factor not isomorphic to $V_q(\lambda)$ is

$$\lambda' = \begin{cases} \lambda \left(\prod_{l=1}^{k'} \alpha_{n,p,aq^{s+k-1-2(l-1)}}^{-1} \right), & \text{if (i) holds,} \\ \lambda \left(\prod_{i=p+1}^{i_0} \prod_{m=1}^{\lambda(h_i)} \alpha_{i,n,aq^{s_i+\lambda(h_i)-1-2(m-1)}}^{-1} \right) \left(\prod_{m=1}^d \alpha_{p,n,aq^{s_p+\lambda(h_p)-1-2(m-1)}}^{-1} \right), & \text{if (ii) holds,} \end{cases}$$

where $d = k' -_{p+1}|\lambda|$. ◇

This is the main result of this paper. It describes the unique non-obvious irreducible factor of the tensor product of an increasing minimal affinization with a Kirillov-Reshetikhin module associated to ω_n , when it exists, as well as the precise condition for its existence. Duality arguments can be used to obtain similar description for the tensor product for all combinations between an increasing or decreasing minimal affinization and a Kirillov-Reshetikhin module associated to ω_1 or ω_n . We give the precise statements in Section 5 and explain how to obtain the other cases from Theorem 2.5.1. Moreover, combining these duality arguments with the main result of [4], we will also prove the following in Section 5. Let

$$V' = V_q(\varpi) \otimes V_q(\omega).$$

It is well-know that the Grothendieck ring of $\tilde{\mathcal{C}}_q$ is commutative and, hence, Theorem 2.5.1 applies as is to V' in place of V .

Corollary 2.5.2. V and V' are indecomposable. Moreover, if condition (i) of Theorem 2.5.1 holds, we have short exact sequences

$$0 \rightarrow V_q(\lambda') \rightarrow V \rightarrow V_q(\lambda) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow V_q(\lambda) \rightarrow V' \rightarrow V_q(\lambda') \rightarrow 0$$

while we have

$$0 \rightarrow V_q(\lambda) \rightarrow V \rightarrow V_q(\lambda') \rightarrow 0 \quad \text{and} \quad 0 \rightarrow V_q(\lambda') \rightarrow V' \rightarrow V_q(\lambda) \rightarrow 0$$

if condition (ii) holds. ◇

Remark 2.5.3. We chose to write Theorem 2.5.1 for a Kirillov-Reshetikhin module associated to ω_n since this makes the notation of this paper closer to that needed for its application to the classification of minimal affinizations of type D . It will be clear from the proof of the theorem that the pair (p, k') is unique if it exists. In fact, this is already obvious in the case that condition (ii) holds. The uniqueness in the case of (i) is proved after (4.3.6) below. Moreover, it follows from (4.3.5) that conditions (i) and (ii) cannot be simultaneously satisfied. One easily checks that the equation in condition (i) of Theorem 2.5.1 is equivalent to saying that

$$\omega_{p,aq^{s_p},\lambda(h_p)} \omega_{n,aq^{s-(k'-1)},k-(k'-1)}$$

corresponds to a (necessarily decreasing) minimal affinization. Similarly, the one in condition (ii) is equivalent to saying that

$$\omega_{p, aq^{sp+p+1}|\lambda|-(k'-1), p|\lambda|-(k'-1)} \omega_{n, aq^s, k}$$

corresponds to a (necessarily increasing) minimal affinization. Moreover, in this case,

$$\left(\prod_{i < p} \omega_{i, aq^{s_i}, \lambda(h_i)} \right) \omega_{p, aq^{sp+p+1}|\lambda|-(k'-1), p|\lambda|-(k'-1)} \omega_{n, aq^s, k}$$

corresponds to a minimal affinization as well. \diamond

3. CHARACTER THEORY

In this section, we present the main tool we will use to prove Theorem 2.5.1: the notion of qcharacter of a representation of $U_q(\tilde{\mathfrak{g}})$ introduced by Frenkel and Reshetikhin in [14] (see also [5]). More precisely, we shall use Nakajima's tableaux description of the qcharacters [25].

3.1. Characters and qCharacters. Let $\mathbb{Z}[P]$ be the integral group ring over P and denote by $e : P \rightarrow \mathbb{Z}[P], \lambda \mapsto e^\lambda$, the inclusion of P in $\mathbb{Z}[P]$ so that $e^\lambda e^\mu = e^{\lambda+\mu}$. The character of an object V from \mathcal{C}_q is defined by

$$\text{ch}(V) = \sum_{\mu \in P} \dim(V_\mu) e^\mu.$$

For $\lambda \in P^+$, let $m_\lambda(V)$ be the multiplicity of $V_q(\lambda)$ as a simple factor of V . It is well-known that the numbers $m_\mu(V)$ can be computed from $\text{ch}(V)$ and vice-versa.

Similarly, for an object V from $\tilde{\mathcal{C}}_q$ and $\omega \in \mathcal{P}^+$, let $m_\omega(V)$ be the multiplicity of $V_q(\omega)$ as a simple factor of V . We now turn to the concept which plays a role analogous to character for the category $\tilde{\mathcal{C}}_q$. It was introduced in [14] under the name of qcharacter. In particular, one can compute the multiplicities $m_\omega(V)$ from the qcharacter of V .

Let $\mathbb{Z}[\mathcal{P}]$ be the integral group ring over \mathcal{P} . Given $\chi \in \mathbb{Z}[\mathcal{P}]$, say

$$\chi = \sum_{\mu \in \mathcal{P}} \chi(\mu) \mu,$$

we identify it with the function $\mathcal{P} \rightarrow \mathbb{Z}, \mu \mapsto \chi(\mu)$. Conversely, any function $\mathcal{P} \rightarrow \mathbb{Z}$ with finite support can be identified with an element of $\mathbb{Z}[\mathcal{P}]$. The qcharacter of $V \in \tilde{\mathcal{C}}_q$ is the element $\text{qch}(V)$ corresponding to the function

$$\mu \mapsto \dim(V_\mu).$$

We shall denote by $\text{wt}_\ell(\chi)$ the support of $\chi \in \mathbb{Z}[\mathcal{P}]$. In particular, we set

$$\text{wt}_\ell(V) = \text{wt}_\ell(\text{qch}(V)) = \{\mu \in \mathcal{P} : V_\mu \neq 0\}.$$

Given an ℓ -weight module V and a vector subspace W of V , let $W_\mu = W \cap V_\mu$. We shall say that W is an ℓ -weight subspace of V if

$$W = \bigoplus_{\mu \in \mathcal{P}} W_\mu.$$

In that case, we set

$$\text{qch}(W) = \sum_{\mu \in \mathcal{P}} \dim W_\mu \mu \quad \text{and} \quad \text{wt}_\ell(W) = \text{wt}_\ell(\text{qch}(W)).$$

Although the tensor product of ℓ -weight vectors is not an ℓ -weight vector in general, we still have the following result [14, Lemma 2]:

Proposition 3.1.1. For every $V, W \in \tilde{\mathcal{C}}_q$, $\text{qch}(V \otimes W) = \text{qch}(V)\text{qch}(W)$. \diamond

3.2. Tableaux and ℓ -Weights. In this subsection we review Nakajima's description of ℓ -weights in terms of tableaux [25]. Fix $a \in \mathbb{C}^\times$ and set

$$(3.2.1) \quad Y_{i,r} := \omega_{i,aq^r} \quad \text{and} \quad A_{i,r} := \alpha_{i,aq^{r-1}}, \quad i \in I, r \in \mathbb{Z}.$$

In particular,

$$(3.2.2) \quad A_{i,r} = Y_{i,r-1}Y_{i,r+1}Y_{i-1,r}^{-1}Y_{i+1,r}^{-1},$$

where we set $Y_{0,r} = Y_{n+1,r} = 1$ for convenience. We also introduce the following notation. Given $i, j \in I, i \leq j, r \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}$, define

$$Y_{i,r,m} = \prod_{k=0}^{m-1} Y_{i,r+2k} = \omega_{i,aq^{m+r-1},m},$$

$$(3.2.3) \quad A_{i,j,r} = \prod_{k=i}^j A_{k,r+k-i+1} \quad \text{and} \quad A_{j,i,r} = \prod_{k=i}^j A_{j+i-k,r+k-i+1}.$$

Remark 3.2.1. The reason for defining $A_{i,r} := \alpha_{i,aq^{r-1}}$ instead of simply $A_{i,r} := \alpha_{i,aq^r}$ is to match with the notation of [14]. The notation $Y_{i,s}$ and $A_{i,s}$, originally used in [14], is more commonly used in the literature about qcharacters and this is the reason we switch to this notation. Note that, if ω is as in (2.5.1), then

$$(3.2.4) \quad \omega = \prod_{i \leq i_0} Y_{i,r_i,\lambda(h_i)} \quad \text{where} \quad r_i = s_i - \lambda(h_i) + 1.$$

In particular, for all $i \leq i_0$ we have

$$(3.2.5) \quad r_i = r_{i_0} - 2|\lambda|_{i_0-1} + i - i_0.$$

\diamond

Consider the fundamental representation $V_q(Y_{1,s}) = V_q(\omega_{1,aq^s})$. Its qcharacter is known to be given by

$$(3.2.6) \quad \text{qch}(V_q(Y_{1,s})) = Y_{1,s} \left(1 + \sum_{j=1}^n A_{1,j,s}^{-1} \right) = \sum_{i=0}^n Y_{i,s+i+1}^{-1} Y_{i+1,s+i}.$$

Represent the element $Y_{i-1,s+i}^{-1} Y_{i,s+i-1}$, $i = 1, \dots, n+1$, by the picture \boxed{i}_s . Given such a box \boxed{i}_s , we shall refer to i as the content of the box and to s as its support. Thus, $\text{wt}_\ell(V_q(Y_{1,s}))$ can be described by the following graph

$$\boxed{1}_s \xrightarrow{1,s+1} \boxed{2}_s \xrightarrow{2,s+2} \dots \xrightarrow{n,s+n} \boxed{n+1}_s$$

where the label $(i, s+i)$ on the i -th arrow indicates that $\boxed{i+1}_s$ is obtained from \boxed{i}_s by multiplication by $A_{i,s+i}^{-1}$. Then, (3.2.6) can be re-written as

$$\text{qch}(V_q(Y_{1,s})) = \sum_{i=1}^{n+1} \boxed{i}_s.$$

Let $\mathbf{B} = \{1, \dots, n+1\}$ equipped with the usual ordering $<$ coming from \mathbb{Z} . Given $k, s \in \mathbb{Z}, k > 0$, a column tableau T of length k with support starting in s is a map

$$T : \{1, \dots, k\} \rightarrow \mathbf{B} \times \mathbb{Z}$$

such that, if we denote by $T(j)_2$ the \mathbb{Z} -component of $T(j)$, then

$$(3.2.7) \quad T(j)_2 = s + 2(k - j) \quad \text{for all } j = 1, \dots, k.$$

We represent T by the picture:

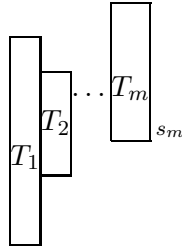
$$(3.2.8) \quad \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array} \quad \text{where} \quad i_j = T(j)_1$$

and $T(j)_1$ denotes the \mathbf{B} -component of $T(j)$. For notational convenience, we set $T(0)_1 = 0$. Notice that we can think of this picture as a vertical juxtaposition of the boxes $\boxed{i_j}_{s+2(k-j)}$ with explicit mention of the support of the k -th box only since the others are recovered from it. Given such a tableau, we associate to it an element $\omega^T \in \mathcal{P}$ given by

$$\omega^T = \prod_{j=1}^k \boxed{i_j}_{s+2(k-j)}.$$

Remark 3.2.2. Nakajima's original definition regards T as a map $\mathbb{Z} \rightarrow \mathbf{B} \cup \{0\}$ such that $T(a) = 0$ if and only if $a \notin \{s, s+2, \dots, s+2(k-1)\}$. Thus, in our notation, $T(j)_2$ corresponds to the j -th element of the support of T in Nakajima's notation while $T(j)_1$ is the value it assumes at that element. \diamond

A tableau T is a finite sequence of column tableaux $T = (T_1, T_2, \dots, T_m)$. If T_j has length k_j and support starting at s_j , the shape of T is defined as the sequence $((k_1, s_1), (k_2, s_2), \dots, (k_m, s_m))$. We represent T graphically by ordered horizontal juxtaposition of the associated pictures (3.2.8) in such a way that the boxes with equal support form a horizontal row:



In particular, if the picture is connected, the supports of all boxes have the same parity and can be recovered from the support s_m of the last box of T_m . We associate to a tableau T the element $\omega^T \in \mathcal{P}$ given by

$$\omega^T = \prod_{j=1}^m \omega^{T_j}.$$

Henceforth, we shall only consider tableaux whose associated picture is connected.

A tableau T is said to be column-increasing (or simply increasing) if the contents in each column strictly increase from top to bottom. Note that a column tableau is increasing of length equal to the content of its last box if and only if $T(j)_1 = j$ for all j . In pictures, T is of the form

$$(3.2.9) \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline i \\ \hline \end{array} \quad s$$

for some $i \in \{1, 2, \dots, n+1\}$, $s \in \mathbb{Z}$. If T is such a column tableau, then

$$(3.2.10) \quad \omega^T = Y_{i, s+i-1}.$$

In particular, if T is an increasing column tableau of length $n + 1$, i.e., if T has the form

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n+1 \\ \hline \end{array}_s$$

for some $s \in \mathbb{Z}$, then $\omega^T = 1$. Hence, adding increasing columns of length $n + 1$ to a tableau T does not change ω^T .

Two tableaux T and T' are said to be equivalent if, for all $(i, a) \in \mathbf{B} \times \mathbb{Z}$, we have

$$\#\{j : (i, a) \in \mathcal{I}m(T_j)\} = \#\{j : (i, a) \in \mathcal{I}m(T'_j)\}.$$

In terms of pictures, T' is obtained from T by permuting the contents of the boxes in the same row. It is easy to see that $\omega^T = \omega^{T'}$ if T and T' are equivalent. The converse is not true, but “almost”:

Lemma 3.2.3. [25, Lemma 4.4] Let T and T' be tableaux. The elements ω^T and $\omega^{T'}$ are equal if and only if T and T' become equivalent after adding several increasing column tableaux of length $n + 1$ to T and T' . \diamond

Lemma 3.2.4. [25, Lemma 4.5] Let T be a tableau. Then, $\omega^T \in \mathcal{P}^+$ if and only if T is equivalent to a tableau T' whose columns are of the form (3.2.9). \diamond

We end this subsection presenting the elementary modifications in column tableaux associated to ℓ -roots. Let T be a tableaux of shape (k, s) and suppose $j \in \{1, \dots, k\}$ is such that $i := T(j)_1 \leq n$. Then, given $i \leq i' \leq n$, one easily checks that $\omega^T A_{i, i', s+2(k-j)+i-1}^{-1} = \omega^{T'}$ where T' is obtained from T by replacing the content of the j -th box by $i' + 1$. In pictures:

$$(3.2.11) \quad \begin{array}{|c|} \hline \vdots \\ \hline i \\ \hline \vdots \\ \hline \end{array}_s A_{i, i', s+2(k-j)+i-1}^{-1} = \begin{array}{|c|} \hline \vdots \\ \hline i'+1 \\ \hline \vdots \\ \hline \end{array}_s.$$

3.3. The qCharacters of Minimal Affinizations. We now study the qcharacters of minimal affinizations in terms of tableaux.

A tableau T with shape $((k_1, s_1), (k_2, s_2), \dots, (k_m, s_m))$ is said to be semi-standard if it is column-increasing and satisfies:

- (i) $s_1 \geq s_2 \geq \dots \geq s_m$;
- (ii) $(i, s) \in \mathcal{I}m(T_j)$ and $(i', s - 2) \in \mathcal{I}m(T_{j+1}) \Rightarrow i \geq i'$.

In terms of pictures, the sequences of diagonal contents from left to right and top to bottom are decreasing (not necessarily strictly). Notice that this implies that the sequences of row contents are strictly decreasing. Given a tableau T , we will denote by $\text{STab}(T)$ the set of semi-standard tableau with the same shape as T .

Recall the definition of increasing and decreasing minimal affinizations given after Theorem 2.4.1. Until the end of this subsection we fix $\lambda \in P^+$ and $\omega \in \mathcal{P}^+$ such that $V_q(\omega)$ is a minimal affinization of $V_q(\lambda)$. Suppose first that $V_q(\omega)$ is an increasing minimal affinization and, hence, ω is give by (3.2.4). Using (3.2.10), one easily sees that $\omega = \omega^T$, where $T = (T^n, T^{n-1}, \dots, T^1)$ with T^i omitted if $\lambda(h_i) = 0$ and, otherwise, $T^i = (T_1^i, \dots, T_{\lambda(h_i)}^i)$ with T_j^i column-increasing with length equal the content of its last box and support starting at $r_i + 2(\lambda(h_i) - j) - i + 1$:

$$(3.3.1) \quad T_j^i = \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline i \\ \hline \end{array}_{r_i + 2(\lambda(h_i) - j) - i + 1}$$

Notice that T has $|\lambda|$ columns and

$$(3.3.2) \quad \omega_j^{T^i} = Y_{i, r_i + 2(\lambda(h_i) - j)}.$$

Observe also that, if the support of the j -th column of T starts at s , then that of the $(j+1)$ -th column starts at $s-2$. Indeed, if they are both columns of T^i , this is obvious. Otherwise, consider the last column of T^i and suppose the next column is the first one of T^k . Then,

$$\begin{aligned} (r_i - i + 1) - (r_k + 2(\lambda(h_k) - 1) - k + 1) &= (r_i - (r_k + 2(\lambda(h_k) - 1))) - i + k \\ &\stackrel{(3.2.5)}{=} (i - k + 2) - i + k \\ &= 2. \end{aligned}$$

Moreover, the top of each column is in a row below the top of the previous column because the length of the rows decrease. In pictures, T has the form:

$$(3.3.3) \quad \begin{array}{c} \vdots \\ \square \\ \vdots \\ \square \\ \vdots \\ \square \end{array} \quad \begin{array}{c} \vdots \\ \square \\ \vdots \\ \square \\ \vdots \\ \square \end{array} \quad \cdots \quad \begin{array}{c} \vdots \\ \square \\ \vdots \\ \square \\ \vdots \\ \square \end{array}$$

Similarly, if $V_q(\omega)$ is a decreasing minimal affinization, then $\omega = \omega^T$, where $T = (T^1, T^2, \dots, T^n)$ with T^i omitted if $\lambda(h_i) = 0$ and, otherwise, $T^i = (T_1^i, \dots, T_{\lambda(h_i)}^i)$ with T_j^i as in (3.3.1). Again, T has $|\lambda|$ columns and, if the support of the first box of the j -th column of T is s , then the support of the first box of the $(j+1)$ -th column is $s-2$. This time, the bottom of each column is in a row below the bottom of the previous column and we have a picture of the form:

$$(3.3.4) \quad \begin{array}{c} \square \\ \vdots \\ \square \\ \vdots \\ \square \\ \vdots \\ \square \end{array} \quad \begin{array}{c} \square \\ \vdots \\ \square \\ \vdots \\ \square \\ \vdots \\ \square \end{array} \quad \cdots \quad \begin{array}{c} \square \\ \vdots \\ \square \\ \vdots \\ \square \\ \vdots \\ \square \end{array}$$

Henceforth, when we say that T is the semi-standard tableau such that $\omega^T = \omega$, we mean the tableau we have described above. For the remainder of this subsection T denotes this tableau.

Lemma 3.3.1. If $S \in \text{STab}(T) \setminus \{T\}$, then $\omega^S \notin \mathcal{P}^+$.

Proof. Suppose $\omega^S \in \mathcal{P}^+$. Then, by Lemma 3.2.4, S is equivalent to a tableau having all of its columns of the form (3.2.9). But T is clearly the unique element of $\text{STab}(T)$ with this property. \square

The next theorem describes the qcharacters of minimal affinizations in terms of semi-standard tableaux. For the proof see [16, Theorems 3.8 and 3.10], [22, Corollary 7.6 and Remark 7.4 (i)], and

references therein. We recall that $V \in \tilde{\mathcal{C}}_q$ is said to be ℓ -minuscule (or special) if $\#\text{wt}_\ell(V) \cap \mathcal{P}^+ = 1$ and it is said to be thin (or quasi ℓ -minuscule) if $\dim(V_q(\omega)_\mu) \leq 1$ for all $\mu \in \mathcal{P}$.

Theorem 3.3.2. $V_q(\omega)$ is thin and

$$\text{qch}(V_q(\omega)) = \sum_{S \in \text{STab}(T)} \omega^S.$$

In particular, $V_q(\omega)$ is also ℓ -minuscule. \diamond

Notice that it follows from this theorem that

$$(3.3.5) \quad S, S' \in \text{STab}(T) \quad \Rightarrow \quad \omega^S = \omega^{S'} \quad \text{iff} \quad S = S'.$$

Example 3.3.3. We now make explicit the qcharacter of the Kirillov-Reshetikhin modules associated to $k\omega_n$. Thus, suppose $\varpi = Y_{n,r,k}$ for some r and set $V = V_q(\varpi)$. Note that $T = (T_1, \dots, T_k)$ is the semi-standard tableau where each column T_j has length n , the content of the last box is n , and the shape of T is

$$((n, r+1+2(k-1)-n), \dots, (n, r+3-n), (n, r+1-n)).$$

By Theorem 3.3.2, the ℓ -weights of V are given by elements of $\text{STab}(T)$. Thus, we have the highest ℓ -weight $\varpi = \omega^T$ and all the ℓ -weights in $\text{wt}_\ell(V) \setminus \{\varpi\}$ are obtained from T by changing the contents of the boxes of T without breaking condition of being semi-standard. Consider the case $k = 1$ first. Then, the corresponding semi-standard tableaux are $T_{1,j}, 1 \leq j \leq n+1$, given as follows:

$$\begin{array}{ccccccc} \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline n-2 \\ \hline n-1 \\ \hline n \\ \hline \end{array} & = \omega^{T_{1,n+1}} & \xrightarrow{n,r+1} & \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline n-2 \\ \hline n-1 \\ \hline n+1 \\ \hline \end{array} & = \omega^{T_{1,n}} & \xrightarrow{n-1,r+2} & \begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline n-2 \\ \hline n \\ \hline n+1 \\ \hline \end{array} = \omega^{T_{1,n-1}} \\ r+1-n & & & r+1-n & & & r+1-n \\ \\ & \xrightarrow{n-2,r+3} & \dots & \xrightarrow{2,r+n-1} & = \omega^{T_{1,2}} & \xrightarrow{1,r+n} & = \omega^{T_{1,1}}. \\ & & & \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \vdots \\ \hline n+1 \\ \hline \end{array} & & & \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \vdots \\ \hline n+1 \\ \hline \end{array} \\ & & & r+1-n & & & r+1-n \end{array}$$

Using (3.2.11), one checks that the pair $(j, r+j)$ written over the arrows correspond to the multiplication by $A_{j,r+j}^{-1}$. Therefore,

$$\text{qch}(V) = \sum_{p=1}^{n+1} \omega^{T_{1,p}} = \varpi \left(1 + \sum_{p=1}^n A_{n,p,r}^{-1} \right).$$

For $k > 1$, we first notice that we can do the same sequence of changes on the first column. Suppose we have done j changes on the first column. Then we can do the same type of changes on the second column up to the j -th change and so on. In other words, the ℓ -weights of V are parameterized by the set of partitions $J(n, k) = \{\mathbf{j} = (j_1, j_2, \dots, j_k) : 0 \leq j_k \leq \dots \leq j_2 \leq j_1 \leq n\}$ and the ℓ -weight associated to $\mathbf{j} \in J$ is

$$(3.3.6) \quad \varpi_{\mathbf{j}} = \varpi \prod_{l=1}^k A_{n, n+1-j_l, r+2(k-l)}^{-1},$$

where we use the convention that $A_{n,n+1,s} = 1$ for all s^1 . In terms of fundamental ℓ -weights, we have

$$(3.3.7) \quad \varpi_j = Y_{n,r,k} \prod_{l=1}^k \left(Y_{n,r+2(k-l)}^{-1} Y_{n-j_l+1,r+2(k-l)+j_l+1}^{-1} Y_{n-j_l,r+2(k-l)+j_l} \right)^{1-\delta_{0,j_l}}.$$

◇

3.4. On Irreducible Tensor Products of Minimal Affinizations. We take a short pause in the study of ℓ -weights via tableaux to prove a proposition which, in particular, implies the existence of the number s in the statement of Theorem 2.5.1. Let $\omega, \varpi \in \mathcal{P}^+$ correspond to minimal affinizations. Then, by Theorem 2.4.1, there exist $a, b \in \mathbb{C}^\times, r_i, s_i \in \mathbb{Z}, \lambda, \mu \in P^+$, such that

$$\omega = \prod_{i \in I} \omega_{i,aq^{r_i}, \lambda(h_i)} \quad \text{and} \quad \varpi = \prod_{i \in I} \omega_{i,bq^{s_i}, \mu(h_i)}.$$

Proposition 3.4.1. If $V_q(\omega) \otimes V_q(\varpi)$ is reducible, there exists $s \in \mathbb{Z}$ such that $a/b = q^s$.

Proof. Let $V = V_q(\omega) \otimes V_q(\varpi), \lambda = \omega\varpi$, and suppose $a/b \neq q^s$ for all $s \in \mathbb{Z}$. We claim that $D := \text{wt}_\ell(V) \cap \mathcal{P}^+ = \{\lambda\}$, which clearly implies the proposition. Indeed, any element of D is of the form $\mu\nu$ with $\mu \in \text{wt}_\ell(V_q(\omega))$ and $\nu \in \text{wt}_\ell(V_q(\varpi))$ by Proposition 3.1.1. If $\mu \neq \omega$, then $\mu \notin \mathcal{P}^+$ by Lemma 3.3.1. Then, it follows from Theorem 3.3.2 that there exists $i \in I, r \in \mathbb{Z}$, such that ω_{i,aq^r}^{-1} appears in μ . As $\mu\nu \in \mathcal{P}^+$, it follows that ω_{i,aq^r} must appear in ν . However, Theorem 3.3.2 also implies that, if $\omega_{i,c}$ appears in ν , then $c = bq^t$ for some t , yielding the desired contradiction. □

Remark 3.4.2. It follows that we can assume from now on that the parameter b in Theorem 2.5.1 is of the form aq^s for some s . In fact, it is well-known (see (5.1.3) below) that we can assume without loss of generality that $a = 1$ and we shall do so. We also use $a = 1$ in (3.2.1). ◇

Let us also recall the description of the simple modules as tensor products of Kirillov-Reshetikhin modules in the case $\mathfrak{g} = \mathfrak{sl}_2$. Thus, let i be the unique element of I . Given $\omega \in \mathcal{P}^+$, it is not difficult to see that there exist unique $m > 0, a_j \in \mathbb{C}^\times$, and $r_j \in \mathbb{Z}_{\geq 1}$ such that

$$\omega = \prod_{j=1}^m \omega_{i,a_j,r_j} \quad \text{with} \quad \frac{a_j}{a_l} \neq q^{\pm(r_j+r_l-2p)} \quad \text{for all} \quad j \neq l \quad \text{and} \quad 0 \leq p < \min\{r_j, r_l\}.$$

This decomposition is called the q -factorization of ω . It was proved in [6, Theorem 4.11] that

$$(3.4.1) \quad V_q(\omega) \cong V_q(\omega_{i,a_1,r_1}) \otimes \cdots \otimes V_q(\omega_{i,a_m,r_m}).$$

3.5. Diagram Subalgebras and Sublattices. Some of the next properties of q characters and tableaux that we will describe are related to the technique of restricting to diagram subalgebras. In this subsection, we fix the necessary notation.

By abuse of language, we will refer to any subset J of I as a subdiagram of the Dynkin diagram of \mathfrak{g} . Let \mathfrak{g}_J be the Lie subalgebra of \mathfrak{g} generated by $x_{\alpha_j}^\pm, j \in J$, and define $\mathfrak{n}_J^\pm, \mathfrak{h}_J$ in the obvious way. Let also Q_J be the subgroup of Q generated by $\alpha_j, j \in J$, and $R_J^+ = R^+ \cap Q_J$. Given $\lambda \in P$, λ_J is the restriction of λ to \mathfrak{h}_J^* and let $\lambda^J \in P$ be such that $\lambda^J(h_j) = \lambda(h_j)$ if $j \in J$ and $\lambda^J(h_j) = 0$ otherwise. Diagram subalgebras $\tilde{\mathfrak{g}}_J$ are defined in the obvious way.

Consider also the subalgebra $U_q(\tilde{\mathfrak{g}}_J)$ generated by $k_j^{\pm 1}, h_{j,r}, x_{j,s}^\pm$ for all $j \in J, r, s \in \mathbb{Z}, r \neq 0$. If $J = \{j\}$, the algebra $U_q(\tilde{\mathfrak{g}}_j) := U_q(\tilde{\mathfrak{g}}_J)$ is isomorphic to $U_q(\mathfrak{sl}_2)$. Similarly we define the subalgebra $U_q(\mathfrak{g}_J)$, etc.

¹Note that, for $j_l \neq 0, A_{n,n+1-j_l,r+2(k-l)}$ is given by the second definition in (3.2.3). Thus, the convention here, used when $j_l = 0$, comes from the usual convention for products applied to the second definition in (3.2.3).

For $\omega \in \mathcal{P}$, let ω_J be the associated J -tuple of rational functions and let $\mathcal{P}_J = \{\omega_J : \omega \in \mathcal{P}\}$. Similarly define \mathcal{P}_J^+ . Notice that ω_J can be regarded as an element of the ℓ -weight lattice of $U_q(\mathfrak{g}_J)$. Let $\pi_J : \mathcal{P} \rightarrow \mathcal{P}_J$ denote the map $\omega \mapsto \omega_J$. If $J = \{j\}$ is a singleton, we write π_j instead of π_J . An ℓ -weight $\omega \in \mathcal{P}$ is said to be J -dominant if $\omega_J \in \mathcal{P}_J^+$. Let also $\mathcal{Q}_J \subset \mathcal{P}_J$ (respectively, \mathcal{Q}_J^+) be the subgroup (submonoid) generated by $\pi_J(\alpha_{j,a}), j \in J, a \in \mathbb{C}^\times$. When no confusion arises, we shall simply write $\alpha_{j,a}$ for its image in \mathcal{P}_J under π_J . Let

$$\iota_J : \mathbb{Z}[\mathcal{Q}_J] \rightarrow \mathbb{Z}[\mathcal{Q}],$$

be the ring homomorphism such that $\iota_J(\alpha_{j,a}) = \alpha_{j,a}$ for all $j \in J, a \in \mathbb{C}^\times$. We shall often abuse of notation and identify \mathcal{Q}_J with its image under ι_J . In particular, given $\mu \in \mathcal{P}$, we set

$$\mu \mathcal{Q}_J = \{\mu \alpha : \alpha \in \iota_J(\mathcal{Q}_J)\}.$$

It will also be useful to introduce the element $\omega^J \in \mathcal{P}$ defined by

$$(\omega^J)_j(u) = \omega_j(u) \quad \text{if } j \in J \quad \text{and} \quad (\omega^J)_j(u) = 1 \quad \text{otherwise.}$$

If $\omega \in \mathcal{P}$ is J -dominant for some subdiagram J , set

$$\chi_J(\omega) = \omega \cdot \iota_J(\omega_J^{-1} \text{qch}(V_q(\omega_J))).$$

Proposition 3.5.1. [17, Corollary 3.15] Let $J \subset I$, $\omega \in \mathcal{P}^+$ and suppose $\mu \in \mathcal{P}$ satisfies:

- (i) $\mu \in \text{wt}_\ell(V_q(\omega))$,
- (ii) $\mu \in \mathcal{P}_J^+$,
- (iii) there is no J -dominant $\varpi > \mu$ satisfying $\varpi \in \text{wt}_\ell(V_q(\omega))$ and $\mu \in \text{wt}_\ell(\chi_J(\varpi))$.

Then $\text{wt}_\ell(\chi_J(\mu)) \subseteq \text{wt}_\ell(V_q(\omega))$. ◇

Remark 3.5.2. Notice that taking $\mu = \omega$ in Proposition 3.5.1, it follows that $\text{wt}_\ell(\chi_J(\omega)) \subseteq \text{wt}_\ell(V_q(\omega))$. ◇

3.6. Further Combinatorial Properties of Tableaux. We now collect several technical lemmas on the combinatorics of tableaux.

Suppose T is an increasing column tableau. We say that T has a gap at the j -th box if

$$T(j)_1 - T(j-1)_1 > 1.$$

The number $T(j)_1 - T(j-1)_1 - 1$ will be referred to as the size of the gap. In particular, for $j = 1$, T has a gap of size $i_1 - 1$ at the first row iff $T(1)_1 = i_1 > 1$. Notice also that, if the length of T is n , then it has at most of one gap, necessarily of size 1.

Lemma 3.6.1. Let T be a column increasing tableau of shape (k, s) with a gap. More precisely, suppose $T(j)_1 = l_1$ and $T(j+1)_1 = l_2$ with $1 \leq l_1 < l_2 - 1 \leq n$, for some $j \in \{1, \dots, k-1\}$. Then $Y_{l_1, s+2(k-j)+l_1-1}$ and $Y_{l_2-1, s+2(k-(j+1))+l_2}^{-1}$ appear in ω^T . Moreover, if $j = k-1$, then $Y_{l_2, s+l_2-1}$ also appears in ω^T .

Proof. By hypothesis, T contains the boxes

$$\boxed{l_1}_{s+2(k-j)} = Y_{l_1-1, s+2(k-j)+l_1}^{-1} Y_{l_1, s+2(k-j)+l_1-1}$$

and

$$\boxed{l_2}_{s+2(k-j)-2} = Y_{l_2-1, s+2(k-j)+l_2-2}^{-1} Y_{l_2, s+2(k-j)+l_1-3}.$$

Since $l_1 < l_2 - 1$, the negative power produced by $\boxed{l_2}_{s+2(k-j)-2}$ cannot be canceled with the positive power produced by $\boxed{l_1}_{s+2(k-j)}$ (the box immediately above it). Also, since T is increasing, $T(j')_1 < l_1$ for all $j' < j$ and $T(j'')_1 > l_2$ for all $j'' > j+1$. Thus, there is no other possibility for canceling $Y_{l_2-1, s+2(k-j)+l_2-2}^{-1}$ implying that $Y_{l_2-1, s+2(k-j)+l_2-2}^{-1}$ appears in ω^T . The proof that $Y_{l_1, s+2(k-j)+l_1-1}$ appears in ω^T is similar. The last statement is also proved in the same manner. □

Lemma 3.6.2. Let T be a semi-standard tableau with shape as in (3.3.3) and $(i, s) \in \mathbf{B} \times \mathbb{Z}$. Suppose the box \boxed{i}_s is part of the the j -th column of T . Then:

- (a) The box \boxed{i}_s is not in any other column of T .
- (b) If $\boxed{i-1}_{s+2}$ is a box in T , it must be in the j -th column.
- (c) If $\boxed{i+1}_{s-2}$ is a box in T , it must be in the j -th column.

Proof. We write down the proof of (b) only since the other items are similar. Suppose $\boxed{i-1}_{s+2}$ appears in the $(j+m)$ -th column, $m \geq 1$. Since T' is as in (3.3.3), this column has a box supported at $s-2m$. Since T' is columns increasing, the content c of the box supported at $s-2m$ is at least $i+m > i$. This contradicts the assumption that T' is semi-standard because the box \boxed{i}_s in column j and the box \boxed{c}_{s-2m} in column $j+m$ are in the same diagonal from left to right and top to bottom. Suppose now that $\boxed{i-1}_{s+2}$ is in the $(j-m)$ -th column, $m \geq 1$. This time (3.3.3) implies that this column has a box supported at $s+2m$. Since all columns are increasing, the content c of the box supported at $s+2m$ is at most $i-m < i$. This contradicts the assumption that T' is semi-standard because the box \boxed{c}_{s+2m} in column $j-m$ and the box \boxed{i}_s in column j are in the same diagonal from left to right and top to bottom. \square

Remark 3.6.3. Note that Lemma 3.6.2 implies that the contributions to ω^T coming from each gap of a given column of T as described in Lemma 3.6.1 are not canceled by terms in other columns. \diamond

The next lemma can be easily proved combinatorially and it is also a consequence of Theorem 3.3.2 together with the fact that the Frenkel-Mukhin algorithm applies for computing the qcharacters of minimal affinizations (see [16] and references therein).

Lemma 3.6.4. Let T be a semi-standard tableau such that $V_q(\omega^T)$ is a minimal affinization. Then, for any $T' \in \text{STab}(T)$, there exists $m \geq 0, i_j \in I, s_j \in \mathbb{Z}$, and elements $T_j \in \text{STab}(T), 0 \leq j \leq m$, such $T_0 = T, T_m = T'$ and $\omega^{T_{j+1}} = \omega^{T_j} A_{i_j, s_j}^{-1}$. \diamond

3.7. Right Negativity. Let $\mathcal{P}_{\mathbb{Z}}$ denote the subgroup of \mathcal{P} generated by $Y_{i,s}, i \in I, s \in \mathbb{Z}$, and we similarly define the subgroup $\mathcal{Q}_{\mathbb{Z}}$ of \mathcal{Q} and the monoids $\mathcal{P}_{\mathbb{Z}}^+$ and $\mathcal{Q}_{\mathbb{Z}}^+$. The following concept defined in [13] will be useful in the proof of Theorem 2.5.1. Given $\omega \in \mathcal{P}_{\mathbb{Z}} \setminus \{\mathbf{1}\}$, set

$$(3.7.1) \quad r(\omega) := \max\{s \in \mathbb{Z} : Y_{i,s}^{\pm 1} \text{ appears in } \omega \text{ for some } i \in I\}.$$

Then, ω is said to be right negative if $Y_{i,r(\omega)}$ does not appear in ω for all $i \in I$. Observe that the product of right negative ℓ -weights is a right negative ℓ -weight and a dominant ℓ -weight is not right negative. Observe also that

$$(3.7.2) \quad r(Y_{i,r,k}) = r + 2(k-1) \quad \text{for all } i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}.$$

Example 3.7.1. Return to Example 3.3.3 and recall (3.3.7). Notice that, if $j_l > 0$ for some l , there exists a gap of size 1 at the $(n-j_l+1)$ -th row of the l -th column of the associated semi-standard tableau. Moreover, if $\varpi_j \neq \varpi$, then $j_1 > 0$ which implies $Y_{n+1-j_1, r+2(k-1)+j_1+1}^{-1} Y_{n-j_1, r+2(k-1)+j_1}$ appears in ϖ_j and

$$(3.7.3) \quad r(\varpi_j) = r + 2(k-1) + j_1 + 1.$$

Indeed, since $l_1 < l_2$ implies $j_{l_1} \geq j_{l_2}$, we have $r + 2(k-l_1) + j_{l_1} > r + 2(k-l_2) + j_{l_2}$. \diamond

The following was proved in [24, Theorem 3.2].

Proposition 3.7.2. Let $\omega = Y_{i,r,k}$ for some $i \in I$, $r \in \mathbb{Z}$, and $k \in \mathbb{Z}_{\geq 0}$. Then, all the elements of $\text{wt}_\ell(V_q(\omega)) \setminus \{\omega\}$ are right negative. Moreover, if $\mu \in \text{wt}_\ell(V_q(\omega)) \setminus \{\omega\}$ is such that $r(\mu) \leq r + 2k$, then

$$\mu = Y_{i,r,s} Y_{i,r+2(s+1),k-s}^{-1} \prod_{j: c_{ij}=-1} Y_{j,r+2s+1,k-s} \quad \text{for some } s = 0, \dots, k-1.$$

In particular, $r(\mu) = r + 2k > r(\omega)$. \diamond

4. PROOF OF THE MAIN THEOREM

4.1. The Scheme of the Proof. Fix the notation of Theorem 2.5.1. The scheme of the proof is as follows. First, using the combinatorics of semi-standard tableaux studied in Section 3.6, we describe the set

$$D := \text{wt}_\ell(V) \cap \mathcal{P}^+$$

and, as a byproduct, we obtain the following proposition.

Proposition 4.1.1. The partial order on \mathcal{P} induces a total order on D and

$$\dim(V_\nu) = 1 \quad \text{for all } \nu \in D.$$

\diamond

This will be done in Sections 4.2 and 4.3. Moving on, we shall see that, if neither of the conditions (i) and (ii) of Theorem 2.5.1 are satisfied, then $\sum_{\nu \in D} \nu$ is part of $\text{qch}(V_q(\lambda))$ and, hence, V is irreducible. Otherwise, we will see that there exists a decomposition of D as a disjoint union

$$D = D_1 \cup D_2$$

such that

- (a) $\sum_{\nu \in D_1} \nu$ is part of $\text{qch}(V_q(\lambda))$;
- (b) $\lambda' \in D_2$ and $\lambda' \notin \text{wt}_\ell(V_q(\lambda))$;
- (c) $\sum_{\nu \in D_2} \nu$ is part of $\text{qch}(V_q(\lambda'))$.

This clearly implies Theorem 2.5.1. For proving these properties of D_1 and D_2 we will use the results on qcharacters that were reviewed in Section 3.7. The aforementioned properties of D will follow from Lemmas 4.4.1 and 4.4.2 below.

Throughout this section we let S be the semi-standard tableau such that $\omega^S = \omega$ and T be the one such that $\omega^T = \varpi$. Recall also from Remark 3.4.2 that we have set $a = a_{i_0} = 1$. We shall work with the expression (3.2.4) for ω and, similarly, we let $r \in \mathbb{Z}$ be such that $\varpi = Y_{n,r,k}$, i.e., $r = s - k + 1$ where s is the number in Theorem 2.5.1 (cf. Proposition 3.4.1). We now rephrase Theorem 2.5.1 in the context of tableaux and in terms of the numbers r_i and r since this is the way we will work in the proof.

Theorem 4.1.2. V has length at most 2 and is reducible if and only if there exist $p \in \text{supp}(\lambda)$ and $k' > 0$ such that either one of the following options hold:

- (i) $k' \leq \min\{\lambda(h_p), k\}$ and $r + 2k + n - p + 2 = r_p + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$ and $r_{i_0} + 2\lambda(h_{i_0}) + n - i_0 + 2 = r + 2k'$, and $p = \max\{i \in I : i|\lambda| \geq k'\}$.

If (i) holds, $\lambda' = \omega \omega^{T'}$ where T' is the only element of $\text{STab}(T)$ whose gaps are all located at the p -th boxes of the first k' columns. If (ii) holds, $\lambda' = \omega^{S'} \varpi$ where S' is obtained from S by replacing the contents of last boxes of the first k' columns by $n + 1$. \diamond

Remark 4.1.3. The formulas for λ' as given in Theorem 2.5.1, or rather in terms of the elements defined in (3.2.1) and (3.2.3), will be obtained as we develop the proof of Theorem 4.1.2. Since we will work with this rephrasing of Theorem 2.5.1, we no longer maintain s and s_i fixed as in Theorem

2.5.1 and allow ourselves to use these symbols for additional local parameters in the several steps of the proof. \diamond

4.2. Fail of Dominance at n for Increasing Minimal Affinizations. As the first step towards describing the set D , we describe the elements of $\text{wt}_\ell(V_q(\omega))$ which are J -dominant where

$$J = I \setminus \{n\}.$$

Recall the definition of the map π_J in Section 3.5 and the definition of right negative elements in Section 3.7.

Lemma 4.2.1. If $\nu \in \text{wt}_\ell(V_q(\omega))$ is not right negative, then $r(\nu) = r_{i_0} + 2(\lambda(h_{i_0}) - 1)$ and $\pi_{i_0}(\nu) = Y_{i_0, r_{i_0}, \lambda(h_{i_0})}$.

Proof. The statement is clear if $\nu = \omega$. Suppose $\nu \neq \omega$ and write $\nu = \omega^{S'}$ with $S' \in \text{STab}(S)$. Since $S' \neq S$, there exists $1 \leq l \leq |\lambda|$ such that the l -th column S'_l of S' has a gap. Assume l is the smallest such index. It follows from Lemma 3.6.1 that $Y_{i, r(\omega^{S'_l})}^{-1}$ appears in $\omega^{S'_l}$ for some $i \in I$. By Lemma 3.6.2, $Y_{i, r(\omega^{S'_l})}$ does not appear in $\omega^{S' \setminus S'_l}$, where $S' \setminus S'_l$ is the tableaux obtained from S' by removing its l -th column. Hence, $Y_{i, r(\omega^{S'_l})}^{-1}$ appears in ν . Since ν is not right negative, there must exist $1 \leq l' \leq |\lambda|$ such that $S'_{l'}$ is gap-free and $r(\omega^{S'_{l'}}) > r(\omega^{S'_l})$. One easily checks that, if $l' > l$ and $S'_{l'}$ has no gaps, then $r(\omega^{S'_{l'}}) < r(\omega^{S'_l})$. Therefore, we must have $1 \leq l' < l$ and, since both S'_1 and $S'_{l'}$ are gap-free, it follows that $r(\omega^{S'_1}) > r(\omega^{S'_{l'}})$ which proves the first statement of the lemma. Since S' is semi-standard and the first column is gap-free, all columns of length i_0 must also be gap-free which implies the second statement. \square

Henceforth, we denote by l_j the length of the j -th column of S . Given $1 \leq c \leq f \leq |\lambda|$ and $0 < p \leq n + 1$, consider the tableau $S_{c, f, p}$ which is the unique tableau in $\text{STab}(S)$ satisfying:

- (1) each column has at most one gap, necessarily at its last box;
- (2) if $j < c$ or $j > f$, the j -th column of $S_{c, f, p}$ does not have a gap;
- (3) if $c \leq j \leq f$, the content of the last box of the j -th column is $\max\{l_j, p\}$.

For notational convenience, we set $S_{c, f, p} = S_{c, |\lambda|, p}$ for $f > |\lambda|$ and $S_{c, f, p} = S$ for $f < c$ as well as for $c > |\lambda|$ and for $p = 0$. Note that the support of the last box of the c -th column of S is

$$s_c = r_{l_c} + 2(l_c |\lambda| - c) - l_c + 1$$

Then, Lemma 3.6.1 implies that

$$(4.2.1) \quad \omega^{S_{c, f, p}} = \left(\prod_{j=0}^{f-c} Y_{l_{c+j}-1, s_c-2j+l_{c+j}} Y_{p-1, s_c+p-2j}^{-1} Y_{p, s_c+p-2j-1} \right) \left(\prod_{i=l_c+1}^n Y_{i, r_i, \lambda(h_i)} \right) \\ \times \left(\prod_{i=1}^{l_f-1} Y_{i, r_i, \lambda(h_i)} \right) \left(\prod_{\substack{j < c: \\ l_j = l_c}} Y_{l_c, r_{l_c} + 2(l_c |\lambda| - j)} \right) \left(\prod_{\substack{j > f: \\ l_j = l_f}} Y_{l_f, r_{l_f} + 2(l_f |\lambda| - j)} \right).$$

The terms in the first parenthesis of (4.2.1) are the ones corresponding to the columns of $S_{c, f, p}$ which are not equal to those of S , while the remaining terms come from the columns which were not modified. By Remark 3.6.3, there are no cancelations in (4.2.1). Here is the main result of this subsection.

Proposition 4.2.2. The J -dominant elements of $\text{wt}_\ell(V_q(\omega))$ are $\omega^{S_{1, f, n+1}}$, $0 \leq f \leq |\lambda|$.

Proof. It is clear from (4.2.1) that $\omega^{S_{1,f,n+1}}$ is J -dominant for all $0 \leq f \leq |\lambda|$. For the converse, by Theorem 3.3.2, any element of $\text{wt}_\ell(V_q(\omega))$ can be represented by an element of $\text{STab}(S)$. Let $S' \in \text{STab}(S)$ and suppose its l -th column has a gap in a box whose content is i . By Lemma 3.6.1 and Remark 3.6.3, it follows that $Y_{i-1,r}^{-1}$ appears in $\omega^{S'}$ and, hence, $i = n+1$ and the gap must be in the last box of the column. This immediately implies that $S' = S_{1,f,n+1}$ where f is the number of the last column having $n+1$ as the content of its last box. \square

We shall need some extra information about the elements $\omega^{S_{c,f,p}}$. Given $1 \leq j \leq |\lambda|$, set

$$(4.2.2) \quad d_j = j - l_{j+1}|\lambda|.$$

Thus, the j -th column is the d_j -th one of length l_j . Notice that, by definition of $S_{c,f,p}$ and (3.2.11), we have

$$(4.2.3) \quad \omega^{S_{c,f,p}} = \omega^{S_{c,f-1,p}} A_{l_f, p-1, r_{l_f} + 2(\lambda(h_{l_f}) - d_f)}^{-1}$$

for all $1 \leq c \leq f \leq |\lambda|, l_c < p \leq n+1$. Iterating this, we get

$$(4.2.4) \quad \omega^{S_{c,f,p}} = \omega \left(\prod_{j=c}^f A_{l_j, p-1, r_{l_j} + 2(\lambda(h_{l_j}) - d_j)}^{-1} \right).$$

Remark 4.2.3. Note that the element S' in Theorem 4.1.2 is $S_{1,k',n+1}$. The formula for λ' in case (ii) of Theorem 2.5.1 is then easily deduced from (4.2.4). \diamond

The following formulas are easily obtained from (4.2.1) (note that, if $l_f = n$, then $i_0 = n$).

$$(4.2.5) \quad \pi_n(\omega^{S_{1,f,n+1}}) = \begin{cases} Y_{n, r_{i_0} + 2(\lambda(h_{i_0}) - f + 1) + n - i_0, f}^{-1}, & \text{if } l_f < n, \\ Y_{n, r_n, \lambda(h_n) - f} Y_{n, r_n + 2(\lambda(h_n) - f + 1), f}^{-1}, & \text{if } l_f = n. \end{cases}$$

Let η be so that (4.2.4) reads $\omega^{S_{c,f,p}} = \omega \eta$. A careful inspection of (4.2.4) and (3.2.3) shows that η can be written in the form

$$(4.2.6) \quad \eta = \prod_{\xi \in \Xi} A_{\xi}^{-1} \quad \text{for some } \Xi = \Xi_{c,f,p} \subseteq I \times \mathbb{Z}.$$

Moreover, (2.2.3) implies that such Ξ is unique. Notice also that, if

$$s = \min\{t : (i, t) \in \Xi \text{ for some } i\}$$

(such minimum is afforded by the pair $(l_f, r_{l_f} + 2(\lambda(h_{l_f}) - d_f) + 1)$), then $s - l_f$ is the support of the last box of the f -th column of S . Moreover, if $c = \min\{j : l_j < p\}$, then

$$(4.2.7) \quad \max\{i : (i, t) \in \Xi \text{ for some } t\} = p - 1,$$

$$(p - 1, s + p - l_f - 1) \in \Xi, \quad \text{and} \quad (i, s + p - l_f - 1) \notin \Xi \quad \text{for } i \neq p - 1.$$

4.3. Dominant ℓ -Weights in V . We now give the description of D and prove Proposition 4.1.1. As in Section 4.2, we set $J = I \setminus \{n\}$. We will consider separately the following two subcases

$$(4.3.1) \quad r_{i_0} + 2\lambda(h_{i_0}) \leq r + 2k$$

and

$$(4.3.2) \quad r + 2k \leq r_{i_0} + 2\lambda(h_{i_0}).$$

Assume first that (4.3.1) holds.

Lemma 4.3.1. The elements of D are of the form $\nu \varpi$ with $\nu \in \text{wt}_\ell(V_q(\omega))$.

Proof. Let $\nu \in \text{wt}_\ell(V_q(\omega))$ and $\mu \in \text{wt}_\ell(V_q(\varpi))$ be such that $\nu\mu \in \mathcal{P}^+$. In particular, $\nu\mu$ is not right negative. Suppose by contradiction that $\mu \neq \varpi$. Then, by Proposition 3.7.2, μ is right negative and it follows from (3.7.3) that

$$r(\mu) = r + 2(k-1) + j + 1$$

for some $1 \leq j \leq n$. Since the product of right negative elements is again right negative, ν is not right negative and Lemma 4.2.1 implies that

$$r(\nu) = r_{i_0} + 2(\lambda(h_{i_0}) - 1).$$

Together with (4.3.1), this implies that $r(\nu) < r(\mu)$. It then follows that $\nu\mu$ is right negative, yielding the desired contradiction. \square

Note that, since $V_q(\omega)$ is thin by Theorem 3.3.2, Lemma 4.3.1 and Proposition 3.1.1 imply the second statement of Proposition 4.1.1 in the present case. We shall now prove that

$$(4.3.3) \quad D = \{\omega^{S_{1,f,n+1}}\varpi : f = 0, 1, \dots, k'\}$$

where $0 \leq k' \leq k$ is either zero or given by the following relation (cf. condition (ii) in Theorem 4.1.2):

$$(4.3.4) \quad r_{i_0} + 2\lambda(h_{i_0}) + n - i_0 + 2 = r + 2k'.$$

Indeed, since the elements in D are of the form $\nu\varpi$ with $\nu \in \text{wt}_\ell(V_q(\omega))$, it follows that ν must be J -dominant and, hence, by Proposition 4.2.2, we must have $\nu = \omega^{S_{1,f,n+1}}$ for some $0 \leq f \leq |\lambda|$. It now easily follows from (4.2.5) that $\omega^{S_{1,f,n+1}}\varpi \in \mathcal{P}^+$ if and only if $0 \leq f \leq k' \leq k$.

Notice that the first statement of Proposition 4.1.1 follows easily from (4.3.3) and (4.2.3), which completes the proof of Proposition 4.1.1 when (4.3.1) holds. Before moving on, notice that equality in (4.3.1) implies that there is no $k' > 0$ satisfying (4.3.4). Indeed, if such k' existed, we would have

$$(4.3.5) \quad 2k' = r_{i_0} + 2\lambda(h_{i_0}) + n - i_0 + 2 - r \stackrel{(4.3.1)}{=} n - i_0 + 2 + 2k > 2k$$

which contradicts $k' \leq k$. This shows that conditions (i) and (ii) of Theorem 2.5.1 cannot be simultaneously satisfied.

From now on till the end of this subsection, assume (4.3.2) holds.

Lemma 4.3.2. If $\mu \in \text{wt}_\ell(V_q(\omega))$ is such that $\mu\varpi \in D$, then $\mu = \omega$.

Proof. Obviously, if $\mu\varpi \in D$, μ must be J -dominant. Let $s = r_{i_0} + 2\lambda(h_{i_0}) + n - i_0$. By (4.2.5), $Y_{n,s}^{-1}$ appears in all J -dominant ℓ -weights of $\text{wt}_\ell(V_q(\omega))$ except ω . We claim that, if $\mu \neq \omega$, then $Y_{n,s}^{-1}$ appears in $\mu\varpi$, which proves the lemma. By definition of ϖ , $Y_{n,r'}$ appears in ϖ iff $r' = r + 2(j-1)$ for some $j = 1, \dots, k$. Since

$$r + 2(k-1) \stackrel{(4.3.2)}{\leq} r_{i_0} + 2(\lambda(h_{i_0}) - 1) < s,$$

the claim follows. \square

Suppose there exists $p \in \text{supp}(\lambda)$ and $k' \in \{1, \dots, \lambda(h_p)\}$ satisfying (cf. condition (i) in Theorem 4.1.2):

$$(4.3.6) \quad r + 2k + n - p + 2 = r_p + 2k'.$$

Observe that the pair (p, k') is unique, if it exists. Indeed, assume (p, k') and (p', k'') satisfy (4.3.6). If $p = p'$ we must obviously have $k' = k''$. Otherwise, without loss of generality, assume that

$$p - p' < 0.$$

To obtain a contradiction, observe that (3.2.5) implies

$$r_{p'} - (r_p + 2(\lambda(h_p) - 1)) \geq p' - p + 2$$

(equality holds if $p' = \min\{i \in \text{supp}(\lambda) : i > p\}$). It follows that,

$$\begin{aligned} p - p' &= r + 2(k - 1) + n - p' + 2 - (r + 2(k - 1) + n - p + 2) \\ &\stackrel{(4.3.6)}{=} r_{p'} + 2(k'' - 1) - (r_p + 2(k' - 1)) \geq r_{p'} + 2(1 - 1) - (r_p + 2(\lambda(h_p) - 1)) \\ &\geq p' - p + 2 > 0. \end{aligned}$$

If a pair (p, k') satisfying (4.3.6) does not exist, we set $p = k' = 0$. Henceforth, we assume that either (p, k') satisfies (4.3.6) or $p = k' = 0$.

If $p \neq 0$, given $0 \leq m \leq k$, consider the tableau $T_{m,p}$ which is the unique tableau in $\text{STab}(T)$ satisfying:

- (1) if $j > m$, the j -th column of $T_{m,p}$ does not have gaps;
- (2) each of the first m columns of $T_{m,p}$ has a gap;
- (3) all gaps occur at the p -th box of the corresponding column.

In particular, $T_{0,p} = T$ and, for convenience, we set $T_{m,n+1} = T$ and $T_{m,p} = T_{k,p}$ for $m > k$. In the spirit of (3.3.6),

$$(4.3.7) \quad T_{m,p} \text{ corresponds to the partition } \mathbf{j} \text{ given by } j_i = \begin{cases} n + 1 - p, & \text{if } i \leq m, \\ 0, & \text{if } i > m. \end{cases}$$

Note that, if $p \neq 0$, (3.2.11) implies

$$(4.3.8) \quad \omega^{T_{m+1,p}} = \omega^{T_{m,p}} A_{n,p,r+2(k-m-1)}^{-1} \quad \text{for all } 0 \leq m < k.$$

Iterating we get

$$(4.3.9) \quad \omega^{T_{m,p}} = \varpi \prod_{l=1}^m A_{n,p,r+2(k-l)}^{-1} \quad \text{for all } 1 \leq m \leq k.$$

Remark 4.3.3. Note that the element T' in Theorem 4.1.2 is $T_{k',p}$. The formula for $\boldsymbol{\lambda}'$ in case (i) of Theorem 2.5.1 is then easily deduced from (4.3.9). \diamond

Set

$$(4.3.10) \quad c = \min\{j : l_j < p\} = 1 + p|\lambda|$$

and consider the subset D' of $\text{wt}_\ell(V)$ defined as follows.

$$\mu \in D' \Leftrightarrow \mu = \begin{cases} \omega \omega^{T_{m,p}}, & \text{for } 0 \leq m \leq k' \\ \omega^{S_{c,f,p}} \omega^{T_{m,p}}, & \text{for } k' \leq m \leq k, f \leq |\lambda|, f = c + m - k' - \epsilon \text{ with } \epsilon \in \{0, 1\}. \end{cases}$$

We will show that

$$(4.3.11) \quad D = D'.$$

Together with (4.3.8) and (4.2.3), (4.3.11) easily implies the first statement of Proposition 4.1.1. Notice that (4.3.11) implies that, if $p = k' = 0$, then $D = \{\boldsymbol{\lambda}\}$. However, we will prove that this is true independently as part of the proof of (4.3.11) (see Proposition 4.3.5).

We begin the proof of (4.3.11) by investigating the elements in D of the form $\omega \boldsymbol{\nu}$ with $\boldsymbol{\nu} \in \text{wt}_\ell(V_q(\varpi))$. Recall from Example 3.3.3 that the elements of $\text{wt}_\ell(V_q(\varpi))$ are in bijection with the set $J(n, k) = \{\mathbf{j} = (j_1, j_2, \dots, j_k) : 0 \leq j_k \leq \dots \leq j_2 \leq j_1 \leq n\}$. Let \mathbf{j} and T' be the tuple and tableaux associated to $\boldsymbol{\nu}$, respectively. In particular, if $\boldsymbol{\nu} \neq \varpi$, we have $j_1 > 0$ and Lemma 3.6.1

implies that the first column of T' contributes with the appearance of the factor $Y_{i_1, r+2k+n-i_1}^{-1}$ in ν , where

$$(4.3.12) \quad i_1 := n - j_1 + 1.$$

This implies that $Y_{i_1, r+2(k-1)+n-i_1+2}$ must appear in ω . Recalling that

$$\omega = \prod_{i \in I} Y_{i, r_i, \lambda(h_i)} = \prod_{i \in I} \prod_{l=1}^{\lambda(h_i)} Y_{i, r_i+2(l-1)},$$

it follows that there exists $1 \leq l \leq \lambda(h_{i_1})$ such that

$$r + 2(k-1) + n - i_1 + 2 = r_{i_1} + 2(l-1).$$

In other words, the pair (i_1, l) satisfies (4.3.6) and, hence, $i_1 = p$ and $l = k'$. In particular, we have shown the following lemma.

Lemma 4.3.4. If $p = k' = 0$ and $\nu \in \text{wt}_\ell(V_q(\varpi))$ is such that $\omega\nu \in D$, then $\nu = \varpi$. \diamond

Proposition 4.3.5. If $p = k' = 0$, then $D = \{\lambda\}$.

Proof. By Lemma 4.3.2, there is no element in D of the form $\mu\varpi$ with $\mu \in \text{wt}_\ell(V_q(\omega)) \setminus \{\omega\}$. Suppose $\mu\nu \in D$ with $\mu \in \text{wt}_\ell(V_q(\omega))$ and $\nu \in \text{wt}_\ell(V_q(\varpi)) \setminus \{\varpi\}$. As before, let j and T' be the tuple and tableaux associated to ν , respectively, and set $i = i_1$ as defined in (4.3.12). Then, applying Lemma 3.6.1 to the first column of T' as before, we see that the terms

$$(4.3.13) \quad Y_{i, s+i-1}^{-1} \quad \text{and} \quad Y_{i-1, s+i-2} \quad \text{appear in } \nu,$$

where

$$s = r + 2(k-i) + n + 1.$$

It then suffices to show that the first listed factor in (4.3.13) cannot be canceled by a factor of μ . Let $S' \in \text{STab}(S)$ be such that $\omega^{S'} = \mu$ and observe that, for canceling that term, S' should have the box \boxed{i}_s and, if $\boxed{i'}_{s-2}$ is also a box in S' , then $i' \neq i+1$. Suppose the box \boxed{i}_s occurred at the l -th column of S' . Assume first that it were the last box of the column. We have two cases:

- (i) the l -th column of S' has length i ;
- (ii) the l -th column of S' has length strictly smaller than i .

We will get a contradiction in both cases. By working with the other columns of T' similarly to how we deduced (4.3.13), we see that

$$(4.3.14) \quad \text{if } Y_{j, s'} \text{ appears in } \nu, \quad \text{then } j \geq i-1 \quad \text{and} \quad s' \leq s+i-2.$$

Case (i). Since S' has the same shape of S , the columns of length i have their last box supported at $r_i + 2j - i - 1$, for $j = 1, \dots, \lambda(h_i)$. Thus, there must exist j such that

$$s = r_i + 2j - i - 1, \quad \text{which implies } r + 2k + n - i + 2 = r_i + 2j$$

and, hence, (i, j) satisfies (4.3.6), contradicting the hypothesis of the proposition.

Case (ii). In this case, the l -th column of S' must have a gap. Suppose that $\boxed{j}_{s'}$ is a box of the l -th column of S' such that $\boxed{j-1}_{s'+2}$ is not a box of this column. Evidently, $j \leq i$ and $s' \geq s$. By Lemma 3.6.1 and Remark 3.6.3, $Y_{j-1, s'+j}^{-1}$ appears in $\omega^{S'}$. Since $\mu\nu \in D$, $Y_{j-1, s'+j}$ must appear in ν and, therefore, $j = i$ by the first assertion in (4.3.14). This implies $s' = s$ and, hence, $s' + j = s + i$. However, the second assertion in (4.3.14) implies that $s' + j < s + i$, yielding the desired contradiction.

Finally, assume that \boxed{i}_s is not the last box of the l -th column of S' . This implies that there exists $j > i+1$ such that \boxed{j}_{s-2} is a box in this column. By Lemma 3.6.1 and Remark 3.6.3,

$Y_{j-1, s-2+j}^{-1}$ appears in $\omega^{S'}$ and, hence, $Y_{j-1, s-2+j}$ must appear in ν . Since $j > i + 1$, we get $s - 2 + j > s + i - 1$ contradicting the second assertion in (4.3.14) again. \square

Now, suppose (p, k') satisfies (4.3.6) and set

$$(4.3.15) \quad s = r + 2(k - p) + n - 1,$$

i.e.,

$$(4.3.16) \quad s \text{ is the support of the } p\text{-th box of the first column of } T.$$

Then, (4.3.6) implies that

$$(4.3.17) \quad s + 2 \text{ is the support of the last box of the } b\text{-th column of } S$$

where

$$(4.3.18) \quad b = c - k'.$$

Note that, by (4.3.10), the b -th column of S is the k' -th one of length p counted from right to left. Since S has the form (3.3.3), it follows that, for all $1 \leq m \leq k$, if s' is the support of a box in the l -th column S , then

$$(4.3.19) \quad s' \leq s + 2 - 2(m - 1) \Rightarrow l \geq b + m - 1.$$

Observe also that, by (4.3.16), $s - 2(m - 1)$ is the support of the p -th box of the m -th column of T .

Lemma 4.3.6. Every element of D is of the form $\omega^{S'} \omega^{T_{m,p}}$ for some $0 \leq m \leq k$ and $S' \in \text{STab}(S)$.

Proof. We will prove that, if $T' \in \text{STab}(T)$, then

$$(4.3.20) \quad T' \neq T_{m,p} \Rightarrow \omega^{T'} \omega^{S'} \notin \mathcal{P}^+ \text{ for all } S' \in \text{STab}(S).$$

Thus, suppose $T' \neq T_{m,p}$ for all $0 \leq m \leq k$ and, by contradiction, suppose there exists S' such that $\omega^{T'} \omega^{S'} \in \mathcal{P}^+$. Since $T_{0,p} = T$, we must have $T' \neq T$ and, hence, the first column of T' must contain a gap (necessarily of size 1 since the length of the column is n). The hypothesis $T' \neq T_{m,p}$ implies that T' has a column containing a gap not located at its p -th box. Suppose the m' -th column of T' is the first such column and assume the gap occurs at the j -th box. Then, Lemma 3.6.1 and Remark 3.6.3 imply that $Y_{j,r'}^{-1}$ appears in $\omega^{T'}$ where

$$(4.3.21) \quad r' = r + 2(k - m' + 1) + n - j = s + 2(p - m' + 1) - j + 1$$

and, hence, $Y_{j,r'}$ must appear in $\omega^{S'}$. This means that $\boxed{j}_{r'-j+1}$ must be a box in S' . Since this box occurs in only one column of S' by Lemma 3.6.2, it follows that

$$(4.3.22) \quad \boxed{j+1}_{r'-j-1} \text{ is not a box in } S'$$

(otherwise, it would cancel the $Y_{j,r'}$ coming from $\boxed{j}_{r'-j+1}$).

Suppose $j > p$ and notice that this implies that,

$$(4.3.23) \quad \text{if } \boxed{i}_{s'} \text{ is a box in } T' \text{ with } s' > s, \text{ then it is the } i\text{-th box of its column.}$$

Indeed, the condition $s' > s$ implies that this box is among the first $p - 1$ boxes of its column. Recall that, since all columns of T have length n , each column of T' has at most one gap. Hence, the assumption on j implies that all gaps in T' occur either at or after the p -th box of each column. Hence, the content of the boxes of T' supported at s' must be equal to its position in the column. This proves (4.3.23).

Say $\boxed{j}_{r'-j+1}$ is in the l -th column of S' . Since $j > p$, by definition of r' we have

$$r' - j + 1 < r + 2(k - m') + n - 2p + 3 = s - 2(m' - 1) + 2,$$

which, together with (4.3.19), implies that $l \geq b + m' - 1 \geq b$. As S' is semi-standard, its b -th column must have a box whose content is at least j . Since the length of the b -th column is p , this implies that the b -th column of S' has a gap. Suppose the j' -th box of the b -th column of S' has a gap and let d be the content of this box. In particular, since the columns are increasing, we have

$$(4.3.24) \quad d > j'$$

Thus, $\boxed{d}_{s+2(p-j')+2}$ is a box in the b -th column of S' while $\boxed{d-1}_{s+2(p-j')+4}$ is not. This implies that $Y_{d-1, s+2(p-j')+d+2}^{-1}$ appears in $\omega^{S'}$ and hence, $\boxed{d-1}_{s+2(p-j')+4}$ must be a box in T' , say, at its l' -th column. Observe that, $s + 2(p - j') + 4 = s + 2(p - (j' - 2))$ is the support of the $(j' - 2)$ -th box of the first column of T' and, hence, it is also the support of the $(j' - l' - 1)$ -th box of the l' -th column of T' . Moreover, since $s + 2(p - j') + 4 > s$, (4.3.23) implies that $d - 1 = j' - l' - 1$. Therefore,

$$d = j' - l' \leq j',$$

yielding a contradiction with (4.3.24).

Suppose now that $j < p$. Since T' is semi-standard, this implies that $m' = 1$ and (4.3.21) gives that

$$r' - j + 1 = s + 2 + 2(p - j)$$

is the support of the j -th box of the b -th column of S' whose content is at least j (because the columns of S' are increasing). Since S' is semi-standard, this implies that

$$(4.3.25) \quad \boxed{j}_{r'-j+1} \text{ is a box in the } l\text{-th column of } S' \text{ with } l \geq b.$$

We treat the cases $l = b$ and $l > b$ separately.

If $l = b$, (4.3.22) implies that the b -th column of S' has $\boxed{d}_{r'-j-1}$ with $d > j + 1$. Lemma 3.6.1 and Remark 3.6.3 then imply that $Y_{d-1, r'-j+d-1}^{-1}$ appears in $\omega^{S'}$, forcing $\boxed{d-1}_{r'-j+1}$ to be a box in T' . Observe that $r' - j + 1 = s + 2(p - (j - 1))$ is the support of the $(j - 1)$ -th box of the first column of T' and recall that, since this column has a gap in the j -th box, all boxes above it have content equal to its position in the column. In particular, the content of the box supported at $r' - j + 1$ is $j - 1 < d - 1$. Since T' is semi-standard, the content of the boxes of the remaining columns supported at $r' - j + 1$ must be at most $j - 1$ contradicting that $\boxed{d-1}_{r'-j+1}$ is a box in T' .

Finally, suppose $l > b$. Using that S' is semi-standard, (4.3.25) implies that the b -th column of S' contains a box $\boxed{j'}_{r'-j+1+2(l-b)}$ with $j' \geq j$. Since, by (4.3.21)

$$r' - j + 1 + 2(l - b) = s + 2 + 2(p - (j - (l - b)))$$

is the support of the $(j - (l - b))$ -th box of the b -th column of S' by (4.3.17) and $j - (l - b) < j$, it follows that there exists a gap in the b -th column of S' at the j'' -th box for some $j'' \leq j - (l - b) < j$. Let d be the content of this box. In particular, $d > j''$. This time, the usual application of Lemma 3.6.1 and Remark 3.6.3 imply that $\boxed{d-1}_{r'-j+3+2(j-j'')}$ must be a box in T' . Let us show that this is impossible, yielding the desired contradiction. Since

$$r' - j + 3 + 2(j - j'') = s + 2(p - (j'' - 2)),$$

(4.3.16) implies that $r' - j + 3 + 2(j - j'')$ is the support of the $(j'' - 2)$ -th box of the first column of T' , which has a gap at the j -th box. Thus, since $j'' < j$ and the boxes above the j -th have their contents equal to their position in the column, T' has $\boxed{j''-2}_{r'-j+3+2(j-j'')}$ in the first column. As $d > j''$, $\boxed{d-1}_{r'-j+3+2(j-j'')}$ is not in the first column of T' . Since the sequence of contents of each row of T' is decreasing, $\boxed{d-1}_{r'-j+3+2(j-j'')}$ cannot be in any other column of T' as well. \square

For the next step, note that, by specializing (3.3.7) to the partitions described in (4.3.7), we get

$$(4.3.26) \quad \omega^{T_{m,p}} = Y_{p-1,s+p-2(m-1),m} Y_{p,s+p-2m+3,m}^{-1} Y_{n,s-2(k-p)-n+1,k-m}.$$

Lemma 4.3.7. If $0 \leq m \leq k$ and $S' \in \text{STab}(S)$ are such that $\omega^{S'} \omega^{T_{m,p}} \in D$, then $S' = S_{c,f,p}$ for some $f \leq |\lambda|$.

Proof. If $S' = S$, then $S' = S_{c,f,p}$ for any $f < c$ and there is nothing to do. Thus, assume $S' \neq S$. It follows from Lemma 4.3.2 that we must also have $T_{m,p} \neq T$, i.e., $m \geq 1$. Observe that, if $p = 1$, then $S_{c,f,p} = S$ since, in this case, $f \leq |\lambda| < c$.

We need to study the structure of gaps in S' . Since each gap contributes with the appearance of a term of the form $Y_{i,r'}^{-1}$ in $\omega^{S'}$ for some r' , it follows that $Y_{i,r'}$ must appear in $\omega^{T_{m,p}}$. By (4.3.26), $Y_{i,r'}$ must be among the following elements:

$$(4.3.27) \quad Y_{p-1,r+2(k-m')+n-p+1}, \quad 1 \leq m' \leq m, \quad \text{and} \quad Y_{n,r+2m''}, \quad 0 \leq m'' < k - m.$$

In particular, $i \in \{p-1, n\}$.

Let us show that we must have $i = p-1$ (in particular, it follows that each column of S' can have at most one gap). Indeed, suppose $i = n$ and that the corresponding gap occurs at the l -th column of S' , necessarily at the last box whose content is then $n+1$. Since S' is semi-standard of the form (3.3.3), this implies that the contents of all the last boxes of the columns to the left of the l -th column are also equal to $n+1$. In particular, if $j \leq \min\{l, b\}$, the content of the last box of the j -th column of S' is $n+1$. Moreover, (4.3.17) and (3.3.3) imply that the support of this box is $s+2(b-j+1)$. Using Lemma 3.6.1, this implies that $Y_{i,r'} = Y_{n,r+2(k+b-j+n-p+1)}$, which contradicts (4.3.27) since $k+(b-j)+(n-p)+1 \geq k+1 > k-m$.

Since, as we have observed, each column of S' has at most one gap, it remains to show that there exists $f \leq |\lambda|$ such that the j -th column of S' has a gap iff $c \leq j \leq f$, the gap occurs at the last box and its size is $p-l_j$.

We start by showing that, if the j -th column of S' has a gap, then $j \geq c$. Assume, by contradiction, that $j < c$. Let s' be the support of the box where the gap occurs. In particular, since the support of the last box of this column is $s+2(b-j+1)$ by (4.3.17), we have $s' \geq s+2(b-j+1)$. Assume first that $j \leq b$. Lemma 3.6.1 and Remark 3.6.3 imply that $Y_{p-1,s'+p}^{-1}$ appears in $\omega^{S'}$. But

$$s' + p \geq s + 2(b-j+1) + p \stackrel{(4.3.15)}{=} r + 2(k+b-j) + n - p + 1 > r + 2(k-1) + n - p + 1,$$

which contradicts (4.3.27). In particular, the b -th column of S' does not have a gap. If $b < j < c$, then $l_j = l_b = p$ (see the comment after (4.3.18)). Since, S' is semi-standard, this implies that the b -th column also has a gap, yielding the desired contradiction.

Next, we show that if the j -th column has a gap, it must occur at its last box. Indeed, Lemma 3.6.1 and (4.3.27) imply that the content of this box must be p . Hence, if this were not the last box, it would follow that the content of the last box is at least $p+1$. Since S' is semi-standard of the form (3.3.3), this would imply that the last box of the b -th column is at least $p+1$ implying that the b -th column would have a gap (because it has p boxes), yielding a contradiction. The same reasoning implies that, if $c \leq j' \leq j$, the last box of the j' -th column has content p and, hence, has a gap (because its length is at most $p-1$). Notice also that, since there is no other gap in the j -th column, all the boxes but the last one must have content equal to their position in the column. In particular, the content of its (l_j-1) -th box is l_j-1 and, hence, the size of the gap is $p-l_j$, as claimed. \square

In light of Lemmas 4.3.6 and 4.3.7, in order to complete the proof of (4.3.11), it remains to check that

$$(4.3.28) \quad \omega^{S_{c,f,p}} \omega^{T_{m,p}} \in \mathcal{P}^+ \iff \begin{array}{l} \text{either } 0 \leq m \leq k' \text{ and } f < c \text{ or } k' \leq m \leq k, \\ f \leq |\lambda| \text{ and, } f = c + m - k' - \epsilon \text{ with } \epsilon \in \{0, 1\}. \end{array}$$

But, for c as defined in (4.3.10) and noting that $l_f < p$, if s_c is as in (4.2.1), then (4.3.17) implies that $s_c = s - 2(k' - 1)$ and we can re-write (4.2.1) as

$$(4.3.29) \quad \omega^{S_{c,f,p}} = \left(\prod_{j=0}^{f-c} Y_{l_{c+j}-1, s-2(k'-1+j)+l_{c+j}} Y_{p-1, s+p-2(k'-1+j)}^{-1} Y_{p, s+p-2(k'-1+j)-1} \right) \\ \times \left(\prod_{\substack{j>f: \\ l_j=l_f}} Y_{l_f, r_{l_f}+2(\lambda(h_{l_f})-d_j)} \right) \left(\prod_{i=p}^n Y_{i, r_i, \lambda(h_i)} \right) \left(\prod_{i=1}^{l_f-1} Y_{i, r_i, \lambda(h_i)} \right).$$

Now, a simple comparison of (4.3.29) with (4.3.26) proves (4.3.28).

It remains to prove the second statement of Proposition 4.1.1, which is clear in the case that $D = \{\lambda\}$. Thus, we can assume (4.3.6) holds. Fix $\mu \in D$, say

$$\mu = \omega^{S_{c,f,p}} \omega^{T_{m,p}}$$

with m and f as in (4.3.28). Suppose $S' \in \text{STab}(S), T' \in \text{STab}(T)$ are such that

$$\omega^{S'} \omega^{T'} = \mu.$$

Since $V_q(\omega)$ and $V_q(\varpi)$ are thin, we are left to show that $S' = S_{c,f,p}$ and $T' = T_{m,p}$. For doing this, notice that, working with (4.3.9) similarly to how we deduced (4.2.6) and (4.2.7), it follows that

$$\omega^{S_{c,f,p}} = \omega \prod_{\xi \in \Xi_1} A_\xi^{-1} \quad \text{and} \quad \omega^{T_{m,p}} = \varpi \prod_{\xi \in \Xi_2} A_\xi^{-1}$$

with $\Xi_1, \Xi_2 \subseteq I \times \mathbb{Z}$ satisfying

$$(4.3.30) \quad (i, t) \in \Xi_1 \Rightarrow i < p$$

while

$$(4.3.31) \quad (i, t) \in \Xi_2 \Rightarrow i \geq p \quad \text{and} \quad t \leq s + p.$$

Note that (4.3.30) is a rephrasing of (4.2.7). Setting $\Xi = \Xi_1 \cup \Xi_2$, it then follows from (2.2.3) that there must exist a partition $\Xi = \Xi'_1 \cup \Xi'_2$ such that

$$\omega^{S'} = \omega \prod_{\xi \in \Xi'_1} A_\xi^{-1} \quad \text{and} \quad \omega^{T'} = \varpi \prod_{\xi \in \Xi'_2} A_\xi^{-1}.$$

We will show that

$$(4.3.32) \quad \Xi_j \subseteq \Xi'_j \quad \text{for } j = 1, 2,$$

which clearly completes the proof of Proposition 4.1.1.

We first show (4.3.32) for $j = 2$. By contradiction, suppose there exists $(i, t) \in \Xi_2 \cap \Xi'_1$. By (3.2.11), this implies that, for obtaining S' from S , a modification $\boxed{i}_{t-i} \rightarrow \boxed{i+1}_{t-i}$ was performed in some column of S . Since $t - i \leq s$ by (4.3.31), (4.3.17) implies that this modification can only be performed in columns to the right of the b -th column of S . In particular, the content of the last box of the b -th column of S' is equal to p . We claim that the above modification also implies that the content of the last box of $(b+1)$ -th column of S' is at least $p+1$, contradicting the fact that S' is semi-standard. Indeed, since $i \geq p$ by (4.3.31), the last box of the modified column has content

larger or equal to $p + 1$. The condition on diagonals for semi-standard tableaux implies that the same holds for the last box of the $(b + 1)$ -th column of S' as claimed.

To prove (4.3.32) for $j = 1$, it suffices to show that we actually have $\Xi_2 = \Xi'_2$. Suppose $\Xi_2 \subsetneq \Xi'_2$ and, hence, $\Xi_1 \cap \Xi'_2 \neq \emptyset$. By Lemma 4.3.6, there exists $0 \leq m' \leq k$ such that $T' = T_{m',p}$ and, hence, (4.3.31) is valid for Ξ'_2 in place of Ξ_2 (it is valid for $T_{l,p}$ for any l). It follows that there exists $(i, t) \in \Xi_1$ with $i \geq p$ contradicting (4.3.30). Hence, $m = m'$ showing that $\Xi_2 = \Xi'_2$.

4.4. The Simple Factors of V . We now complete the proof of Theorem 4.1.2. If neither (4.3.4) nor (4.3.6) hold, then D is a singleton by (4.3.3) and (4.3.11) and, hence, V is irreducible.

Assume first that (4.3.4) holds and set

$$\mu_j = \varpi \omega^{S_{1,j,n+1}} \quad \text{for } 0 \leq j \leq k'.$$

Recall that $S_{1,j,n+1} = S_{1,|\lambda|,n+1}$ for all $j \geq |\lambda|$. Thus, $D = \{\mu_j : 0 \leq j \leq \min\{k', |\lambda|\}\}$ by (4.3.3), while (4.2.3) implies that $\mu_{j+1} \leq \mu_j$. In particular,

$$(4.4.1) \quad \lambda' = \mu_{k'} = \min D.$$

Under the present assumption, as explained in Section 4.1, Theorem 4.1.2 follows from the second statement of Proposition 4.1.1 together with the following lemma.

Lemma 4.4.1. We have:

- (a) $\mu_j \in \text{wt}_\ell(V_q(\lambda))$ for all $0 \leq j \leq \min\{|\lambda|, k' - 1\}$. In particular, if $k' > |\lambda|$, $D \subseteq \text{wt}_\ell(V_q(\lambda))$.
- (b) $\mu_{k'} \notin \text{wt}_\ell(V_q(\lambda))$ if $k' \leq |\lambda|$.

Proof. Since $\mu_0 = \lambda$, part (a) clearly holds for $j = 0$. For $j > 0$, consider

$$\mu'_j = \omega^{S_{1,j,n}} \varpi$$

and observe that $S_{1,j,n} = S$ if $l_j = n$. We claim that

$$\mu'_j \in \text{wt}_\ell(V_q(\lambda)) \quad \text{for all } j > 0,$$

which is obvious if $l_j = n$. Thus, assume $l_j < n$. Using Remark 3.5.2 with λ in place of ω and $J = I \setminus \{n\}$, we have $\text{wt}_\ell(\chi_J(\lambda)) \subseteq \text{wt}_\ell(V_q(\lambda))$ and, hence, it suffices to show that

$$(4.4.2) \quad \mu'_j \in \text{wt}_\ell(\chi_J(V_q(\lambda))).$$

By definition of $S_{1,j,n}$, all of its columns of length smaller than n coincide with the corresponding column of S and, hence, $\lambda_J = \omega_J$. Therefore, $V_q(\lambda_J)$ is an increasing minimal affinization whose highest weight corresponds to the tableau S' obtained from S by removing the columns of length n . Note that the j -th column of S becomes the j' -th column of S' where $j' = j - \lambda(h_n)$. Theorem 3.3.2 then implies that $(\omega^{S'_{1,j',n}})_J \in \text{wt}_\ell(V_q(\lambda_J))$. One easily checks that

$$\lambda \cdot \iota_J \left((\omega_J^{-1} \omega^{S'_{1,j',n}})_J \right) = \mu'_j$$

completing the proof of (4.4.2).

In particular, it follows that, for all j , μ'_j satisfies condition (i) of Proposition 3.5.1 with λ in place of ω . We show next that, setting $J = \{n\}$, conditions (ii) and (iii) of Proposition 3.5.1 are also satisfied by μ'_j and, therefore,

$$(4.4.3) \quad \text{wt}_\ell(\chi_{\{n\}}(\mu'_j)) \subseteq \text{wt}_\ell(V_q(\lambda)).$$

Proceeding as in the proof of (4.2.5), one sees that

$$(4.4.4) \quad \pi_n(\omega^{S_{1,j,n}}) = \begin{cases} Y_{n,r_{i_0}+2(\lambda(h_{i_0})-j)+n-i_0,j}, & \text{if } l_j < n, \\ Y_{n,r_n,\lambda(h_n)}, & \text{if } l_j = n, \end{cases}$$

which implies condition (ii). Condition (iii) is obvious if $l_j = n$. If $l_j < n$, (4.2.4) applied to $S_{1,j,n}$ implies that there exists $\Xi \subseteq (I \setminus \{n\}) \times \mathbb{Z}$ such that

$$\mu'_j = \lambda \prod_{\xi \in \Xi} A_\xi^{-1}.$$

Since, for every $\{n\}$ -dominant ν , the elements of $\text{wt}_\ell(\chi_{\{n\}}(\nu))$ are of the form $\nu \prod_{r \in \mathbb{Z}} A_{n,r}^{-m_r}$ with $m_r \in \mathbb{Z}_{\geq 0}$, condition (iii) follows.

Part (a) of the lemma now follows if we show that

$$(4.4.5) \quad \mu_j \in \text{wt}_\ell(\chi_{\{n\}}(\mu'_j)) \quad \text{for all } 1 \leq j \leq \min\{|\lambda|, k' - 1\}.$$

Observe that, since $S_{1,j,n+1}$ is obtained from $S_{1,j,n}$ by modifying the contents of the last boxes of the first j columns from n to $n+1$, (3.2.11) together with (3.2.5) and (4.3.4) implies that

$$(4.4.6) \quad \mu_j = \mu'_j \left(\prod_{l=1}^j A_{n,r+2(k'-l)-1}^{-1} \right).$$

Assume first that $l_j < n$ and observe that

$$\pi_n(\mu'_j) \stackrel{(4.4.4)}{=} Y_{n,r_{i_0}+2(\lambda(h_{i_0})-j)+n-i_0,j} Y_{n,r,k} \stackrel{(4.3.4)}{=} Y_{n,r+2(k'-1-j),j} Y_{n,r,k}.$$

One easily checks that, if $j \leq \min\{|\lambda|, k' - 1\}$, the above is the q -factorization of $\pi_n(\mu'_j)$. Thus, (3.4.1) implies that

$$V_q(\pi_n(\mu'_j)) \cong V_q(Y_{n,r+2(k'-1-j),j}) \otimes V_q(Y_{n,r,k}).$$

one now deduces (4.4.5) by applying (3.3.6) to the first factor of this tensor product and comparing with (4.4.6). Indeed, the ℓ -roots in (4.4.6) coincide with those appearing in (3.3.6) for the constant non zero partition (i.e., for the lowest weight).

If $l_j = n$, which implies $\mu'_j = \lambda, j \leq \lambda(h_n) \leq |\lambda|$, and $i_0 = n$, then

$$\pi_n(\lambda) \stackrel{(4.4.4)}{=} Y_{n,r_n,\lambda(h_n)} Y_{n,r,k} \stackrel{(4.3.4)}{=} Y_{n,r-2(\lambda(h_n)-k'+1),\lambda(h_n)} Y_{n,r,k}$$

and this is the q -factorization of $\pi_n(\lambda)$ provided $\lambda(h_n) \leq k' - 1$ (in particular, $j \leq k' - 1$). One then deduces (4.4.5) as in the previous case (but μ_j may arise from a weight higher than the lowest one in $V_q(Y_{n,r-2(\lambda(h_n)-k'+1),\lambda(h_n)})$). Finally, if $\lambda(h_n) > k' - 1$, then

$$\pi_n(\lambda) = Y_{n,r_n,\lambda(h_n)+k-k'+1} Y_{n,r,k'-1}$$

is the q -factorization of $\pi_n(\lambda)$ and, hence

$$V_q(\pi_n(\lambda)) \cong V_q(Y_{n,r_n,\lambda(h_n)+k-k'+1}) \otimes V_q(Y_{n,r,k'-1}).$$

This time, we apply (3.3.6) to the second factor of this tensor product to obtain μ_j for $j \leq k' - 1$, thus proving (4.4.5).

To prove part (b), it suffices to show that there exist two simple modules V' and V'' such that

$$(4.4.7) \quad \mu_{k'} \notin \text{wt}_\ell(V' \otimes V'') \quad \text{and} \quad V_q(\lambda) \text{ is a simple factor of } V' \otimes V''.$$

For doing this, let $V' = V_q(\omega^{S'})$ where S' is the tableau formed by the first $k' - 1$ columns of S . In particular, V' is an increasing minimal affinization. Evidently, $\lambda(\omega^{S'})^{-1} = \varpi \omega^{S \setminus S'} \in \mathcal{P}^+$ and, therefore, letting V'' be the simple module having this as its highest ℓ -weight, the second claim in (4.4.7) is satisfied.

To complete the proof of (4.4.7), we begin by observing that V'' is also an increasing minimal affinization. More precisely, we show that the tableau S'' obtained by the juxtaposition of T and $S \setminus S'$ is of the form (3.3.3). Note that the hypothesis $k' \leq |\lambda|$ implies that $S \setminus S' \neq \emptyset$ and that the support of the last box of its first column (the k' -th column of S) is $r_{l_{k'}} - l_{k'} + 1 + 2(\lambda(h_{l_{k'}}) - d_{k'})$

(recall (4.2.2)). On the other hand, the support of the last box of the last column of T is $r - n + 1$ and, hence, we need to show that

$$(r - n + 1) - 2 = r_{l_{k'}} - l_{k'} + 1 + 2(\lambda(h_{l_{k'}}) - d_{k'}).$$

But this is easily checked using (3.2.5) and (4.3.4).

Using (4.2.4) as before, we see that there exists $\Xi \subseteq I \times \mathbb{Z}$ such that

$$\mu_{k'} = \lambda \prod_{\xi \in \Xi} A_{\xi}^{-1}.$$

Thus, (4.4.7) follows if we show that there is no partition $\Xi = \Xi' \cup \Xi''$ such that

$$\omega^{S'} \prod_{\xi \in \Xi'} A_{\xi}^{-1} \in \text{wt}_{\ell}(V') \quad \text{and} \quad \omega^{S''} \prod_{\xi \in \Xi''} A_{\xi}^{-1} \in \text{wt}_{\ell}(V'').$$

By contradiction, assume such a partition exists. It follows from (3.2.11) and Lemma 3.6.4 that each element A_{ξ}^{-1} , $\xi \in \Xi$, corresponds to adding 1 to the content of a particular box of some element of either $\text{STab}(S')$ or $\text{STab}(S'')$ so that the new tableau remains semi standard. More precisely, if $\xi = (i, l)$, then (3.2.11) implies that the modification associated to A_{ξ}^{-1} is of the form

$$\boxed{i}_{l-i} \longrightarrow \boxed{i+1}_{l-i}.$$

Inspecting (4.2.4), one checks that

$$\max\{l - i : (i, l) \in \Xi\} = r_{i_0} + 2\lambda(h_{i_0}) - i_0 - 1 =: s.$$

Note that (3.2.5) and (4.3.4) imply that

$$(4.4.8) \quad s + 2 \text{ is the support of the last box of the } (k - k' + 1)\text{-th column of } T$$

(which is the k' -th one counted from right to left). Let $\xi = (i, l)$ be such that $l - i = s$. This means that the boxes of S' and S'' supported above s are not being modified. Together with (4.4.8), it follows that the first $k - k' + 1$ columns of S'' are not modified. In particular, since the first k columns of S'' come from T and $V_q(\omega^{S''})$ is a minimal affinization, all first k columns of S'' are not modified.

On the other hand, another inspection of (4.2.4) shows that

$$(n, s - 2(k' - 1) + n) \in \Xi.$$

This corresponds to a modification of the form

$$\boxed{n}_{s-2(k'-1)} \longrightarrow \boxed{n+1}_{s-2(k'-1)},$$

in some column of some element of $\text{STab}(S') \cup \text{STab}(S'')$. Since the content is n , this must be the last box the column. Since the modified tableau is semi standard, the last box of each column to the left must also have $n + 1$ as content. Since the first columns of S'' are left unmodified, it follows that $(n, s - 2(k' - 1) + n) \notin \Xi''$. Observing that, by construction, the box of S' having the lowest support is supported at $s - 2(k' - 1) + 2$, it follows that $(n, s - 2(k' - 1) + n)$ cannot be in Ξ' as well, yielding the desired contradiction. \square

It remains to prove Theorem 4.1.2 when (4.3.6) holds. In that case, by (4.3.11),

$$D = \left\{ \omega^{S_{c,c+m-k'-1,p}} \omega^{T_{m,p}}, \omega^{S_{c,c+m-k',p}} \omega^{T_{m,p}} : 0 \leq m \leq \min\{k, |\lambda|_{p-1} + k'\} \right\},$$

where $c = {}_p|\lambda| + 1$ (recall that $S_{c,f,p} = S$ for $f < c$ and for $c > |\lambda|$, $S_{c,f,p} = S_{c,|\lambda|,p}$ for $f > |\lambda|$, and $T_{m,p} = T_{k,p}$ for $m > k$). Let

$$\mu_m := \omega \omega^{T_{m,p}} \quad \text{for } 0 \leq m \leq k$$

and observe that $\lambda' = \mu_{k'}$ and

$$\{\nu \in D : \nu \geq \lambda'\} = \{\mu_m : 0 \leq m \leq \min\{k', k\}\}.$$

In particular, if either $k' > k$ or $p = \min \text{supp}(\lambda)$, then $\lambda' = \mu_k$ is the smallest element of D . As explained in Section 4.1, Theorem 4.1.2 follows from the second statement of Proposition 4.1.1 together with the following lemma.

Lemma 4.4.2. We have

- (a) $\{\nu \in D : \nu > \lambda'\} \subseteq \text{wt}_\ell(V_q(\lambda))$.
- (b) If $k' > k$, then $\lambda' \in \text{wt}_\ell(V_q(\lambda))$ or, equivalently, $D \subseteq \text{wt}_\ell(V_q(\lambda))$.
- (c) If $k' \leq k$, then $\lambda' \notin \text{wt}_\ell(V_q(\lambda))$.
- (d) If $p > \min \text{supp}(\lambda)$, then $\{\omega^{S_{c,c+m-k',p}} \omega^{T_{m,p}} : k' \leq m \leq \min\{k, |\lambda|_{p-1} + k'\}\} \subseteq \text{wt}_\ell(V_q(\lambda))$.
- (e) If $k' \leq k$, then $\{\omega^{S_{c,c+m-k'-1,p}} \omega^{T_{m,p}} : k' \leq m \leq \min\{k, |\lambda|_{p-1} + k'\}\} \subseteq \text{wt}_\ell(V_q(\lambda'))$.

Proof. The proof of part (c) is similar in spirit to that of part (b) of the previous lemma, but more complicated. Similarly, the proofs of the other parts have the same spirit of that of part (a) of the previous lemma. In particular, the arguments will contain the proof of a chain of claims of the form

$$\nu \in \text{wt}_\ell(\chi_J(V_q(\mu_J))) \subseteq \text{wt}_\ell(V_q(\pi))$$

for some $\pi \in \mathcal{P}^+$, $J \subseteq I$ connected, $\mu \in \text{wt}_\ell(V_q(\pi)) \cap \mathcal{P}_J^+$, and $\nu \in \mathcal{P}$. At the beginning of each part we will give a summary of the associated chain by drawing a picture of the form

$$\pi \longrightarrow \mu \xrightarrow{J} \nu$$

For most arrows we will have to check that the hypothesis of Proposition 3.5.1 applies with π in place of ω . Note that, after part (b), there is nothing to prove in part (d) if $k' > k$. Hence, we will assume without further mention that $k' \leq k$ once parts (a) and (b) are proved.

Parts (a) and (b): Since $\mu_0 = \lambda$, we have $\mu_0 \in \text{wt}_\ell(V_q(\lambda))$. Given $1 \leq m \leq k$, define

$$\mu'_m = \omega \omega^{T_{m,p+1}}.$$

The summary of the proof is given by the picture

$$\lambda \xrightarrow{\{p+1, \dots, n\}} \mu'_m \xrightarrow{\{p\}} \mu_m$$

We begin by proving that

$$(4.4.9) \quad \mu'_m \in \text{wt}_\ell(V_q(\lambda)) \quad \text{for all } 1 \leq m \leq k$$

which is obvious if $p = n$ since $T_{m,n+1} = T$. Thus, assume $p < n$. Using Remark 3.5.2 with λ in place of ω and $J = \{p+1, \dots, n\}$, we have $\text{wt}_\ell(\chi_J(\lambda)) \subseteq \text{wt}_\ell(V_q(\lambda))$ and, hence, in order to prove (4.4.9), it suffices to show that

$$\mu'_m \in \text{wt}_\ell(\chi_J(V_q(\lambda))) \quad \text{for all } 1 \leq m \leq k.$$

Since $p \notin J$, applying Proposition 4.3.5 to the algebra $U_q(\mathfrak{g}_J)$ with ω_J in place of ω and ϖ_J in place of ϖ , we get

$$V_q(\omega_J) \otimes V_q(\varpi_J) \cong V_q(\lambda_J).$$

The ℓ -weights of $V_q(\varpi_J)$ are also determined by the elements of $\text{STab}(T)$. Therefore, $(\omega^{T_{m,p'}})_J \in \text{wt}_\ell(V_q(\varpi_J))$ for all m, p' . One easily checks that

$$\lambda \cdot {}_\ell J ((\omega^{-1} \varpi^{-1} \omega^{T_{m,p+1}})_J) = \mu'_m$$

completing the proof of (4.4.9).

Next, we use Proposition 3.5.1 with λ in place of ω and $J = \{p\}$. The previous paragraph implies that μ'_m satisfies condition (i) of that Proposition for all m . Equation (4.3.26), with $p+1$

in place of p , implies that condition (ii) is also satisfied. As for condition (iii) with $p < n$ (if $p = n$ this condition is obvious), (4.3.9) implies that

$$\boldsymbol{\mu}'_m = \boldsymbol{\lambda} \cdot \prod_{l=1}^m A_{n,p+1,r+2(k-l)}^{-1}.$$

On the other hand, for any $\boldsymbol{\nu} \in \mathcal{P}_{\{p\}}^+$, the elements of $\text{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\nu}))$ are of the form $\boldsymbol{\nu} \prod_{s \in \mathbb{Z}} A_{p,s}^{-m_s}$ with $m_s \in \mathbb{Z}_{\geq 0}$. Since $p < n$, condition (iii) must also be satisfied. It then follows from Proposition 3.5.1 that

$$\text{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\mu}'_m)) \subseteq \text{wt}_\ell(V_q(\boldsymbol{\lambda})).$$

Parts (a) and (b) of the lemma now follows if we show that

$$(4.4.10) \quad \boldsymbol{\mu}_m \in \text{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\mu}'_m)) \quad \text{for all} \quad 1 \leq m \leq \min\{k' - 1, k\}.$$

Observe that (4.3.9) implies that

$$(4.4.11) \quad \boldsymbol{\mu}_m = \boldsymbol{\mu}'_m \left(\prod_{l=1}^m A_{p,r+2(k-l)+n-p+1}^{-1} \right).$$

Moreover, (4.3.26) implies that

$$\pi_p(\boldsymbol{\mu}'_m) = Y_{p,r+2(k-m)+n-p,m} Y_{p,r_p,\lambda(h_p)} \stackrel{(4.3.6)}{=} Y_{p,r_p+2(k'-1-m),m} Y_{p,r_p,\lambda(h_p)}.$$

One easily checks that, for $m \leq \min\{k' - 1, k\}$, the above is the q -factorization of $\pi_p(\boldsymbol{\mu}'_m)$. Thus, (3.4.1) implies that

$$V_q(\pi_p(\boldsymbol{\mu}'_m)) \cong V_q(Y_{p,r_p+2(k'-1-m),m}) \otimes V_q(Y_{p,r_p,\lambda(h_p)}) = V_q(Y_{p,r+2(k-m)+n-p,m}) \otimes V_q(Y_{p,r_p,\lambda(h_p)}).$$

We now proceed as in the Proof of part (a) of Lemma 4.4.1. Namely, applying (3.3.6) for the subalgebra $U_q(\tilde{\mathfrak{g}}_p)$ to the first factor of the above tensor product and comparing with (4.4.11) one deduces (4.4.10).

Part (c): Consider the tableau S' formed by the first $_{p+1}|\lambda|$ columns of S , the tableau S'' formed by juxtaposing the first $\lambda(h_p) - k' + 1$ columns of length p of S and T , and the tableau S''' formed by the remaining columns of S (i.e., the tableau whose columns are those to the right of the b -th column of S where b is given by (4.3.18)). Then,

$$\begin{aligned} \boldsymbol{\omega}^{S'} &= \prod_{i=p+1}^{i_0} Y_{i,r_i\lambda(h_i)}, & \boldsymbol{\omega}^{S''} &= Y_{p,r_p+2(k'-1),\lambda(h_p)-k'+1} Y_{n,r,k}, \\ \boldsymbol{\omega}^{S'''} &= \left(\prod_{i=1}^{p-1} Y_{i,r_i,\lambda(h_i)} \right) Y_{p,r_p,k'-1}, & \text{and} \quad \boldsymbol{\omega}^{S'} \boldsymbol{\omega}^{S''} \boldsymbol{\omega}^{S'''} &= \boldsymbol{\lambda}. \end{aligned}$$

In particular, $V_q(\boldsymbol{\lambda})$ is the simple quotient of the submodule of $V_q(\boldsymbol{\omega}^{S'}) \otimes V_q(\boldsymbol{\omega}^{S''}) \otimes V_q(\boldsymbol{\omega}^{S'''})$ generated by the top weight space and part (c) follows if we show that

$$(4.4.12) \quad \boldsymbol{\lambda}' \notin \text{wt}_\ell \left(V_q(\boldsymbol{\omega}^{S'}) \otimes V_q(\boldsymbol{\omega}^{S''}) \otimes V_q(\boldsymbol{\omega}^{S'''}) \right).$$

It is clear from the construction of S' and S''' that $V_q(\boldsymbol{\omega}^{S'})$ and $V_q(\boldsymbol{\omega}^{S'''})$ are increasing minimal affinizations. On the other hand, (4.3.6) implies that $V_q(\boldsymbol{\omega}^{S''})$ is a decreasing minimal affinization. In any case, the ℓ -weights of all three factors are represented by the corresponding set of semi-standard tableaux.

By (4.3.9),

$$(4.4.13) \quad \boldsymbol{\lambda}' = \boldsymbol{\lambda} \left(\prod_{l=1}^{k'} A_{n,p,r+2(k-l)}^{-1} \right) = \boldsymbol{\lambda} \prod_{\xi \in \Xi} A_\xi^{-1}$$

for some $\Xi \subseteq I \times \mathbb{Z}$. Equation (4.4.12) follows if we show that there is no partition $\Xi = \Xi' \cup \Xi'' \cup \Xi'''$ such that

$$\omega^{S'} \prod_{\xi \in \Xi'} A_{\xi}^{-1} \in \text{wt}_{\ell}(V_q(\omega^{S'})), \quad \omega^{S''} \prod_{\xi \in \Xi''} A_{\xi}^{-1} \in \text{wt}_{\ell}(V_q(\omega^{S''})), \quad \omega^{S'''} \prod_{\xi \in \Xi'''} A_{\xi}^{-1} \in \text{wt}_{\ell}(V_q(\omega^{S'''})).$$

By contradiction, assume such a partition exists. As before, each element $\xi \in \Xi$, say $\xi = (i, l)$, corresponds to a modification of the form

$$\boxed{i}_{l-i} \longrightarrow \boxed{i+1}_{l-i}$$

in some tableau belonging to $\text{STab}(S') \cup \text{STab}(S'') \cup \text{STab}(S''')$. Inspecting (4.4.13), one checks that

$$(4.4.14) \quad \max\{l - i : i \in I, (i, l) \in \Xi\} = s \quad \text{and} \quad \min\{i : (i, l) \in \Xi \text{ for some } l \in \mathbb{Z}\} = p,$$

where s is given by (4.3.16). This means that the boxes of S' , S'' , and S''' with support larger than s are not modified. Together with (4.3.17), it follows that S' and the columns of S'' coming from S are left unmodified. The condition on diagonals for semi-standard tableaux imply that the (possibly) modified element of $\text{STab}(S'')$ has the first p boxes of every column coinciding with that of S'' .

On the other hand, another inspection of (4.4.13), recalling that $k' \leq k$, shows that

$$\xi_0 := (p, s - 2(k' - 1) + p) \in \Xi.$$

This corresponds to a modification of the form

$$(4.4.15) \quad \boxed{p}_{s-2(k'-1)} \longrightarrow \boxed{p+1}_{s-2(k'-1)}.$$

Since all the tableaux are column increasing, if the box on which this modification is being performed is the j -th box on its column, we must have $j \leq p$. Hence, it follows from the previous paragraph that $\xi_0 \in \Xi'''$. We will show that this is a contradiction.

Indeed, by construction, the last box of the last column of length p of S''' is supported at $s - 2(k' - 1) + 2$. If we had $\min \text{supp}(\lambda) = p$, it would follow that S''' had no box supported at $s - 2(k' - 1)$, yielding a contradiction. Thus, we can assume that $\min \text{supp}(\lambda) < p$ and, hence, the first column of S''' which has a box supported at $s - 2(k' - 1)$ has length $i < p$. This implies that, before the modification (4.4.15) can be performed, we need a modification of the form

$$\boxed{i}_{s-2(k'-1)} \longrightarrow \boxed{i+1}_{s-2(k'-1)}$$

which implies that $(i, s - 2(k' - 1) + i) \in \Xi$, contradicting the second statement in (4.4.14).

Part (d): Fix $k' \leq m \leq \min\{k, |\lambda|_{p-1} + k'\}$, let $f = c + m - k'$, and define

$$\nu = \omega^{S_{c,f,p}} \omega^{T_{m,p}}, \quad \nu' = \omega^{S_{c,f,p}} \omega^{T_{m,p+1}}.$$

The summary of the proof is given by the picture

$$\lambda \xrightarrow{\{p+1, \dots, n\}} \mu'_m \xrightarrow{\{1, \dots, p-1\}} \nu' \xrightarrow{\{p\}} \nu$$

An analogous argument to that used to prove (4.4.9) shows that

$$(4.4.16) \quad \nu' \in \text{wt}_{\ell}(V_q(\lambda)).$$

Indeed, by (4.4.9), $\mu'_m \in \text{wt}_{\ell}(V_q(\lambda))$. Proceeding as in the paragraph after the proof of (4.4.9), this time with $J = \{1, \dots, p-1\}$, we see that

$$(4.4.17) \quad \text{wt}_{\ell}(\chi_J(V_q(\mu'_m))) \subseteq \text{wt}_{\ell}(V_q(\lambda)).$$

Consider the tableaux \tilde{S} obtained from S by restriction to J , i.e., by removing the columns of length larger or equal to p and regarding it as a tableau for the diagram sub-algebra associated to J . Consider also $\tilde{c} = c - p|\lambda|$, $\tilde{f} = f - p|\lambda|$, and observe that

$$(\mu'_m)_J = \omega_J = \omega^{\tilde{S}} \quad \text{and} \quad \omega_J(\omega^{-1}\omega^{S_{c,f,p}})_J = \omega^{\tilde{S}_{\tilde{c},\tilde{f},p}} \in \text{qch}(V_q(\omega_J)).$$

Since

$$\nu' = \mu'_m \cdot {}_{\iota_J}((\omega^{-1}\omega^{S_{c,f,p}})_J) = \mu'_m \cdot {}_{\iota_J}((\mu'_m)^{-1}\omega^{\tilde{S}_{\tilde{c},\tilde{f},p}}) \in \text{wt}_\ell(\chi_J(V_q(\mu'_m))),$$

(4.4.16) follows from (4.4.17).

Next, we use Proposition 3.5.1 with λ in place of ω and $J = \{p\}$. Note that (4.4.16) implies that ν' satisfies condition (i) of that Proposition. Equations (4.3.26), with $p+1$ in place of p , and (4.3.29) imply that condition (ii) is also satisfied. As for condition (iii), note that the condition $m \geq k'$ implies $f \geq c$ and, since $c = p|\lambda| + 1$, it follows that

$$l_j < p \quad \text{for all } j \geq c.$$

In particular, (4.3.9) and (4.2.4) imply that

$$\nu' = \lambda\eta^{-1} \quad \text{with } \eta \in \mathcal{Q}_{J'}^+, \quad J' = I \setminus \{p\}.$$

On the other hand, for any $\mu \in \mathcal{P}_{\{p\}}^+$, the elements of $\text{wt}_\ell(\chi_{\{p\}}(\mu))$ are of the form $\mu \prod_{l \in \mathbb{Z}} A_{p,l}^{-m_l}$, proving that condition (iii) must also be satisfied. It then follows from Proposition 3.5.1 that

$$\text{wt}_\ell(\chi_{\{p\}}(\nu')) \subseteq \text{wt}_\ell(V_q(\lambda))$$

and part (d) follows if we show that

$$(4.4.18) \quad \nu \in \text{wt}_\ell(\chi_{\{p\}}(\nu')).$$

Observe that (4.3.9) implies that

$$(4.4.19) \quad \nu = \nu' \prod_{l=1}^m A_{p,r+2(k-l)+n-p+1}^{-1}.$$

Moreover, (4.3.26) and (4.3.29) imply that

$$\begin{aligned} \pi_p(\nu') &= Y_{p,r+2(k-m)+n-p,m} Y_{p,r_p,\lambda(h_p)} \left(\prod_{l=0}^{m-k'} Y_{p,s+p-2(k'-1+l)-1} \right) \\ &\stackrel{(4.3.15)}{=} Y_{p,r+2(k-m)+n-p,m} Y_{p,r_p,\lambda(h_p)} \prod_{l=0}^{m-k'} Y_{p,r_p-2(l+1)} \\ &= Y_{p,r+2(k-m)+n-p,m} Y_{p,r_p,\lambda(h_p)} Y_{p,r_p+2(k'-1-t),m-k'+1} \\ &= Y_{p,r+2(k-m)+n-p,m} Y_{p,r_p+2(k'-1-m),\lambda(h_p)+m-k'+1}. \end{aligned}$$

The hypothesis $p > \min \text{supp}(\lambda)$ was used in the first equality (for $p = \min \text{supp}(\lambda)$ the term in parenthesis would not exist). Using that $r + 2(k - m) + n - p = r_p + 2(k' - 1 - m)$ (by (4.3.6)), one easily checks that the the last line above is the q -factorization of $\pi_p(\nu')$ and we apply (3.4.1) to get

$$V_q(\pi_p(\nu')) \cong V_q(Y_{p,r+2(k-m)+n-p,m}) \otimes V_q(Y_{p,r_p+2(k'-1-m),\lambda(h_p)+m-k'+1}).$$

Applying (3.3.6) as before to the first factor of the above tensor product and comparing with (4.4.19) one easily deduces (4.4.18).

Part (e): Fix $k' \leq m \leq \min\{k, |\lambda|_{p-1} + k'\}$ and set $f = c + m - k' - 1$ and

$$\nu = \omega^{S_{c,f,p}} \omega^{T_{m,p}}.$$

We need to show that

$$(4.4.20) \quad \nu \in \text{wt}_\ell(V_q(\lambda')).$$

If $m = k'$, then $\nu = \lambda'$ and this is obvious. For $m > k'$, consider

$$\mu = \omega^{S_{c,f,p-1}} \omega^{T_{m,p}}, \quad \nu' = \omega \omega^{T_{m,p}}, \quad \text{and} \quad \nu'' = \omega \omega^{T'},$$

where T' is the tableau whose first k' columns coincide with those of $T_{k',p}$ and the remaining columns are equal to those of $T_{m,p+1}$. The summary of the proof is given by the picture

$$\lambda' \xrightarrow{\{p+1, \dots, n\}} \nu'' \xrightarrow{\{p\}} \nu' \xrightarrow{\{1, \dots, p-2\}} \mu \xrightarrow{\{p-1\}} \nu$$

One easily checks that $T' \in \text{STab}(T)$ and uses (4.3.9) to get

$$(4.4.21) \quad \nu'' = \lambda' \left(\prod_{l=k'+1}^m A_{n,p+1,r+2(k-l)}^{-1} \right) \quad \text{and} \quad \nu' = \nu'' \left(\prod_{l=k'+1}^m A_{p,r+2(k-l)+n-p+1}^{-1} \right).$$

An analogous argument to that used to prove (4.4.9) (with $(\omega^{T_{k',p}})_J = Y_{n,r,k-k'}$ in place of ϖ_J), shows that

$$(4.4.22) \quad \nu'' \in \text{wt}_\ell(\chi_J(V_q(\lambda'))) \subseteq \text{wt}_\ell(V_q(\lambda')) \quad \text{with} \quad J = \{p+1, \dots, n\}.$$

Next, we use Proposition 3.5.1 with λ' in place of ω and $J = \{p\}$. Note that (4.4.22) implies that ν'' satisfies condition (i) of that Proposition. Equation (4.3.26) implies that the term of the form $Y_{p,t}^{-1}$ coming from $T_{k',p}$ in ν'' is the same one that $T_{k',p}$ contributes to λ' . Since λ' is dominant, it means that $Y_{p,t}$ appears in ω . Therefore, $Y_{p,t}^{-1}$ does not appear in ν'' proving that condition (ii) is also satisfied. Condition (iii) follows since the elements of $\text{wt}_\ell(\chi_p(\lambda'))$ are of the form $\lambda' \prod_{l \in \mathbb{Z}} A_{p,l}^{-m_l}$ which is incompatible with the first part of (4.4.21). Thus, it follows from Proposition 3.5.1 that

$$\text{wt}_\ell(\chi_p(\nu'')) \subseteq \text{wt}_\ell(V_q(\lambda')).$$

Observe that

$$\begin{aligned} \pi_p(\nu'') &\stackrel{(4.4.21)}{=} \pi_p(\lambda') \pi_p \left(\prod_{l=k'+1}^m A_{p+1,r+2(k-l)+n-p+1}^{-1} \right) \\ &= Y_{p,r_p+2k', \lambda(h_p)-k'} \left(\prod_{l=k'+1}^m Y_{p,r+2(k-l)+n-p} \right) \\ &= Y_{p,r_p+2k', \lambda(h_p)-k'} Y_{p,r+2(k-m)+n-p, m-k'} \\ &\stackrel{(4.3.6)}{=} Y_{p,r_p+2k', \lambda(h_p)-k'} Y_{p,r_p+2(k'-m-1), m-k'}. \end{aligned}$$

One easily checks that the above is the q -factorization of $\pi_p(\nu'')$ and, hence,

$$\begin{aligned} V_q(\pi_p(\nu'')) &\cong V_q(Y_{p,r_p+2k', \lambda(h_p)-k'}) \otimes V_q(Y_{p,r_p+2(k'-t-1), t-k'}) \\ &= V_q(Y_{p,r_p+2k', \lambda(h_p)-k'}) \otimes V_q(Y_{p,r+2(k-t)+n-p, t-k'}). \end{aligned}$$

Applying (3.3.6) to the first factor of the above tensor product as usual and comparing with the second part of (4.4.21), we get

$$(4.4.23) \quad \nu' \in \text{wt}_\ell(\chi_p(\nu'')) \subseteq \text{wt}_\ell(V_q(\lambda')).$$

We now apply a similar argument to prove that

$$(4.4.24) \quad \text{wt}_\ell(\chi_J(V_q(\nu'))) \subseteq \text{wt}_\ell(V_q(\lambda')) \quad \text{with} \quad J = \{1, \dots, p-1\}.$$

Note that (4.4.23) implies that ν' satisfies condition (i) of Proposition 3.5.1 with λ' in place of ω and J as in (4.4.24). Condition (ii) is clear from (4.3.26). Since the elements of $\text{wt}_\ell(\chi_J(\lambda'))$ are of the form $\lambda' \eta^{-1}$ with $\eta \in \mathcal{Q}_J$ which is incompatible with (4.4.21), condition (iii) is also satisfied and (4.4.24) follows from Proposition 3.5.1.

Since, by (4.3.26),

$$\pi_{\{1,\dots,p-2\}}(\boldsymbol{\nu}') = \prod_{i=1}^{p-2} Y_{i,r_i,\lambda(h_i)} = \pi_{\{1,\dots,p-2\}}(\boldsymbol{\omega}),$$

it follows that $\boldsymbol{\nu}'_{\{1,\dots,p-2\}} = \boldsymbol{\omega}^{\tilde{S}}$, where \tilde{S} is obtained from S by restriction to $\tilde{J} = \{1, \dots, p-2\}$.

Consider also $\tilde{c} = c -_{p-1} |\lambda|$, $\tilde{f} = f -_{p-1} |\lambda|$, and observe that

$$(\boldsymbol{\nu}')_{\tilde{J}} = \boldsymbol{\omega}_{\tilde{J}} = \boldsymbol{\omega}^{\tilde{S}} \quad \text{and} \quad \boldsymbol{\omega}_{\tilde{J}}(\boldsymbol{\omega}^{-1} \boldsymbol{\omega}^{S_{c,f,p}})_{\tilde{J}} = \boldsymbol{\omega}^{\tilde{S}_{\tilde{c},\tilde{f},p}} \in \text{qch}(V_q(\boldsymbol{\omega}_{\tilde{J}})).$$

Since

$$\boldsymbol{\mu} = \boldsymbol{\nu}' \cdot \iota_{\tilde{J}}((\boldsymbol{\omega}^{-1} \boldsymbol{\omega}^{S_{c,f,p}})_{\tilde{J}}) = \boldsymbol{\nu}' \cdot \iota_{\tilde{J}}((\boldsymbol{\nu}')_{\tilde{J}}^{-1} \boldsymbol{\omega}^{\tilde{S}_{\tilde{c},\tilde{f},p}}) \in \text{wt}_{\ell}(\chi_{\tilde{J}}(V_q(\boldsymbol{\nu}'))),$$

we get

$$(4.4.25) \quad \boldsymbol{\mu} \in \text{wt}_{\ell}(\chi_{\tilde{J}}(\boldsymbol{\nu}')) \subseteq \text{wt}_{\ell}(\chi_J(\boldsymbol{\nu}')) \subseteq \text{wt}_{\ell}(V_q(\boldsymbol{\lambda}'))$$

with J as in (4.4.24).

After (4.4.25), (4.4.20) follows if we show that

$$(4.4.26) \quad \text{wt}_{\ell}(\chi_{p-1}(\boldsymbol{\mu})) \subseteq \text{wt}_{\ell}(V_q(\boldsymbol{\lambda}')) \quad \text{and} \quad \boldsymbol{\nu} \in \text{wt}_{\ell}(\chi_{p-1}(\boldsymbol{\mu})).$$

To prove the first claim in (4.4.26), we use Proposition 3.5.1 once more, this time with $\boldsymbol{\lambda}'$ in place of $\boldsymbol{\omega}$ and $J = \{p-1\}$. By (4.4.25), $\boldsymbol{\mu}$ satisfies condition (i) of that Proposition while condition (ii) is easily checked using (4.3.26) and (4.3.29). Note that the first conclusion in (4.4.25) (the “ \in ”), together with (4.4.21), implies that

$$\boldsymbol{\mu} = \boldsymbol{\lambda}' \boldsymbol{\eta}^{-1} \quad \text{with} \quad \boldsymbol{\eta} \in \mathcal{Q}_{I \setminus \{p-1\}}^+.$$

Therefore, (2.2.3) implies that condition (iii) is satisfied. The first claim in (4.4.26) now follows from Proposition 3.5.1.

To prove the second claim in (4.4.26), recall the definitions of $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in terms of tableaux and observe that (4.2.4) together with (3.2.5) implies

$$(4.4.27) \quad \boldsymbol{\nu} = \boldsymbol{\mu} \left(\prod_{l=0}^{m-k'-1} A_{p-1,r_p+2(k'-m+l)}^{-1} \right).$$

As before, we now compute the q -factorization of $\pi_{p-1}(\boldsymbol{\mu})$ and apply the usual argument with tensor products to complete the proof. We split the analysis in two cases according to whether $l_f = p-1$ or $l_f < p-1$. In the former case we have

$$\begin{aligned} \pi_{p-1}(\boldsymbol{\mu}) &\stackrel{(4.3.26)}{=} Y_{p-1,r_{p-1},\lambda(h_{p-1})} \left(\prod_{l=0}^{m-1} Y_{p-1,r+2(k-l)+n-p-1} \right) \\ &\stackrel{(4.3.6)}{=} Y_{p-1,r_{p-1},\lambda(h_{p-1})} \left(\prod_{l=0}^{m-1} Y_{p-1,r_p+2(k'-l-1)-1} \right) \\ &= Y_{p-1,r_{p-1},\lambda(h_{p-1})} \left(\prod_{l=0}^{m-1} Y_{p-1,r_p+2(k'-m+l)-1} \right) \\ &= Y_{p-1,r_{p-1},\lambda(h_{p-1})} \left(\prod_{l=m-k'}^{m-1} Y_{p-1,r_p+2(k'-m+l)-1} \right) \left(\prod_{l=0}^{m-k'-1} Y_{p-1,r_p+2(k'-m+l)-1} \right) \\ &\stackrel{(3.2.5)}{=} Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'} \left(\prod_{l=0}^{m-k'-1} Y_{p-1,r_p+2(k'-m+l)-1} \right) \\ &= Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'} Y_{p-1,r_p+2(k'-m)-1,m-k'}. \end{aligned}$$

One easily checks that the above is the q -factorization of $\pi_{p-1}(\nu')$. Thus, (3.4.1) implies that

$$V_q(\pi_{p-1}(\mu)) \cong V_q(Y_{p-1, r_{p-1}, \lambda(h_{p-1})+k'}) \otimes V_q(Y_{p-1, r_p+2(k'-m)-1, m-k'}).$$

Applying (3.3.6) to the second factor of this tensor product and comparing with (4.4.27) one easily deduces the second claim of (4.4.26) in this case. Finally, if $l_f < p-1$, (4.3.26) and (4.3.29) give

$$\begin{aligned} \pi_{p-1}(\mu) &= Y_{p-1, r_{p-1}, \lambda(h_{p-1})} \left(\prod_{i=l_f+1}^{p-2} \prod_{l=1}^{\lambda(h_i)} Y_{p-1, r_i+2(\lambda(h_i)-l)+p-i-1} \right) \\ &\quad \times \left(\prod_{l=1}^{d_f} Y_{p-1, r_{l_f}+2(\lambda(h_{l_f})-l)+p-l_f-1} \right) \left(\prod_{l=0}^{m-1} Y_{p-1, r+2(k-l)+n-p-1} \right) \\ &\stackrel{(3.2.5)}{=} \left(\prod_{l=0}^{m-k'-1} Y_{p-1, r_p+2(k'-m+l)-1} \right) \left(\prod_{l=0}^{m-1} Y_{p-1, r+2(k-l)+n-p-1} \right) \\ &\stackrel{(4.3.6)}{=} \left(\prod_{l=0}^{m-k'-1} Y_{p-1, r_p+2(k'-m+l)-1} \right) \left(\prod_{l=0}^{m-1} Y_{p-1, r_p+2(k'-l-1)-1} \right) \\ &= \left(\prod_{l=0}^{m-k'-1} Y_{p-1, r_p+2(k'-m+l)-1} \right) \left(\prod_{l=0}^{m-1} Y_{p-1, r_p+2(k'-m+l)-1} \right) \\ &= Y_{p-1, r_p+2(k'-m)-1, m-k'} Y_{p-1, r_p+2(k'-m)-1, m}. \end{aligned}$$

One easily checks that the above is the q -factorization of $\pi_{p-1}(\mu)$ and, hence,

$$V_q(\pi_{p-1}(\mu)) \cong V_q(Y_{p-1, r_{p-1}, \lambda(h_{p-1})+k'+1}) \otimes V_q(Y_{p-1, r_p+2(k-m)-1, t-k'}).$$

by (3.4.1). We are once again done by applying (3.3.6) to the second factor of this tensor product and comparing with (4.4.27). The proof of the second claim of (4.4.26) is complete, as well as the proof of the lemma. \square

5. THE COROLLARIES

In this section, we prove Corollary 2.5.2 and state and prove the other versions of Theorem 2.5.1 concerning tensor products of general minimal affinizations with KR modules supported either at last or the first node of the Dynkin diagram. The proofs will rely most strongly on duality type arguments which we review in the first subsection.

5.1. Duality and Cartan Involution. Given a finite-dimensional $U_q(\mathfrak{g})$ -module V , let V^* denote the dual module defined using the antipode as usual, and similarly for $U_q(\tilde{\mathfrak{g}})$. Given $\lambda \in P^+$ and $\omega \in \mathcal{P}^+$, we have:

$$(5.1.1) \quad \begin{aligned} V_q(\lambda)^* &\cong V_q(\lambda^*) & \text{and} & & V_q(\omega)^* &\cong V_q(\omega^*) \\ & & \text{where} & & \end{aligned}$$

$$\lambda^* = -w_0 \lambda \quad \text{and} \quad \omega_i^*(u) = \omega_{w_0 \cdot i}(q^{-h^\vee} u).$$

Here, $h^\vee = n+1$ is the (dual) Coxeter number of \mathfrak{g} , w_0 is the longest element of \mathcal{W} and $w_0 \cdot i = j$ iff $w_0 \omega_i = -\omega_j$. To simplify notation, we set

$$(5.1.2) \quad \bar{i} = w_0 \cdot i = h^\vee - i.$$

For a proof of the second isomorphism in (5.1.1) see [13].

We will also need the automorphisms given by the following proposition [3, Propositions 1.5 and 1.6].

Proposition 5.1.1.

(a) Given $a \in \mathbb{C}^\times$, there exists a unique Hopf algebra automorphism τ_a of $U_q(\tilde{\mathfrak{g}})$ such that

$$\tau_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm, \quad \tau_a(h_{i,r}) = a^r h_{i,r}, \quad \tau_a(k_i^\pm) = k_i^\pm.$$

(b) There exists a unique algebra automorphism κ of $U_q(\tilde{\mathfrak{g}})$ such that

$$\kappa(x_{i,r}^\pm) = -x_{i,-r}^\mp, \quad \kappa(h_{i,r}) = -h_{i,-r}, \quad \kappa(k_i^{\pm 1}) = k_i^{\mp 1}.$$

Moreover $(\kappa \otimes \kappa) \circ \Delta = \Delta^{\text{op}} \circ \kappa$, where Δ^{op} is the opposite comultiplication of $U_q(\tilde{\mathfrak{g}})$.² \diamond

Given $\omega \in \mathcal{P}^+$, define $\omega^{\tau_a} \in \mathcal{P}^+$ by

$$\omega_i^{\tau_a}(u) = \omega_i(au).$$

One easily checks that the pullback $V_q(\omega)^{\tau_a}$ of $V_q(\omega)$ by τ_a satisfies

$$(5.1.3) \quad V_q(\omega)^{\tau_a} \cong V_q(\omega^{\tau_a}).$$

In particular, we get

$$(5.1.4) \quad (V_q(\omega)^*)^* \cong V_q(\omega)^{\tau_{q^{-2h^\vee}}}.$$

Quite clearly, κ is an involution,

$$(5.1.5) \quad \kappa(\Psi_i^\pm(u)) = \Psi_i^\mp(u), \quad \text{and} \quad \kappa(\Lambda_i^\pm(u)) = (\Lambda_i^\mp(u))^{-1}.$$

The pullback V^κ of an irreducible $U_q(\tilde{\mathfrak{g}})$ -module V by κ is, evidently, an irreducible module as well and we can use (5.1.5) to compute its highest ℓ -weight. Indeed, if $V \cong V_q(\omega)$ with $\omega \in \mathcal{P}^+$, then the lowest-weight vector of V is the highest-weight vector of V^κ . Therefore, if $\varpi \in \text{wt}_\ell(V)$ is the lowest ℓ -weight, we must have, by (5.1.5) and (2.2.2), that

$$(5.1.6) \quad V^\kappa \cong V_q((\varpi^-)^{-1}).$$

Since

$$\varpi_i(u) = (\omega_i(q^{h^\vee} u))^{-1}$$

(see for instance [5, Proposition 3.6]), it follows that

$$\varpi_i^-(u) = (\omega_i^-(q^{-h^\vee} u))^{-1}.$$

In other words,

$$(\varpi^-)^{-1} = (\omega^-)^*$$

and, hence,

$$(5.1.7) \quad V_q(\omega)^\kappa \cong V_q(\omega^\kappa) \quad \text{where} \quad \omega^\kappa = (\omega^-)^*.$$

One easily checks that

$$(\omega^*)^\kappa = \omega^- \quad \text{and} \quad (\omega^\kappa)^* = ((\omega^-)^*)^*$$

which proves that

$$(5.1.8) \quad (V_q(\omega)^*)^\kappa \cong V_q(\omega^-) \quad \text{and} \quad (V_q(\omega)^\kappa)^* \cong (V_q(\omega^-)^*)^* \cong V_q(\omega^-)^{\tau_{q^{-2h^\vee}}}.$$

Note that the maps $\omega \mapsto \omega^-$ and $\omega \mapsto \omega^*$ are monoid endomorphisms of \mathcal{P}^+ . Hence, they induce group endomorphisms of \mathcal{P} which we also denote by $-$ and $*$. Inspired by (5.1.7), we denote by κ the composition $* \circ -$. The following lemma is obvious.

Lemma 5.1.2. The endomorphisms $-$, $*$, and κ of \mathcal{P} are bijective. \diamond

²The automorphism κ is most often denoted by $\hat{\omega}$ in the literature and its restriction to $U_q(\mathfrak{g})$, typically denoted by ω , is referred to as the Cartan automorphism of $U_q(\mathfrak{g})$. We chose to modify the notation to avoid visual confusion with our most often used symbol for a Drinfeld polynomial: ω .

The proof of the next proposition is straightforward from (5.1.1) and (5.1.7) and the definition of ω^- .

Proposition 5.1.3. Let $\omega \in \mathcal{P}^+$, $\lambda = \text{wt}(\omega)$, $V = V_q(\omega)$, and $V^- = V_q(\omega^-)$. The following are equivalent.

- (i) V is an increasing minimal affinization of $V_q(\lambda)$.
- (ii) V^- is a decreasing minimal affinization of $V_q(\lambda)$.
- (iii) V^* is a decreasing minimal affinization of $V_q(-w_0\lambda)$.
- (iv) V^κ is an increasing minimal affinization of $V_q(-w_0\lambda)$. \diamond

The following proposition is easily established by standard arguments.

Proposition 5.1.4. Let $V \in \tilde{\mathcal{C}}_q$ and $a \in \mathbb{C}^\times$. Then, the lengths of V^{τ_a} , V^* , and of V^κ are all equal to that of V . \diamond

Finally, recall that, given a Hopf algebra H , the antipode is an anti-automorphism of the co-multiplication. Using this and the last statement of Proposition 5.1.1, it follows that, for any two finite-dimensional $U_q(\tilde{\mathfrak{g}})$ -modules V and W , we have

$$(5.1.9) \quad (V \otimes W)^* \cong W^* \otimes V^* \quad \text{and} \quad (V \otimes W)^\kappa \cong W^\kappa \otimes V^\kappa.$$

5.2. The General Version of the Main Theorem. Fix $\lambda \in P^+$, ω as in (2.4.2), and set

$$i_0 = \max(\text{supp}(\lambda)), \quad a = a_{i_0}.$$

as before. Thus, there exist $r_i \in \mathbb{Z}$, $i \in I$, such that

$$(5.2.1) \quad \omega_i = \omega_{i, a_i, \lambda(h_i)} = Y_{i, r_i, \lambda(h_i)}.$$

By definition of $Y_{i, r, m}$, we have

$$r_{i_0} = 1 - \lambda(h_{i_0}).$$

If $V_q(\omega)$ is an increasing minimal affinization, then the r_j are related as in (3.2.5). If $V_q(\omega)$ is a decreasing minimal affinization, then

$$(5.2.2) \quad r_i = r_{i_0} + 2 \sum_{i+1} |\lambda|_{i_0} + i_0 - i \quad \text{for all } i \in \text{supp}(\lambda).$$

Fix also

$$\varpi = Y_{e, r, k} \quad \text{for some } r, k \in \mathbb{Z}, k > 0, e \in \{1, n\} \subseteq I,$$

and set

$$\lambda = \omega \varpi, \quad V = V_q(\omega), \quad \text{and} \quad W = V_q(\varpi).$$

Let also $V^- = V_q(\omega^-)$ and $W^- = V_q(\varpi^-)$.

Corollary 5.2.1. The length of $V \otimes W$ is at most 2 and the following are equivalent

- (i) $V \otimes W$ is irreducible.
- (ii) $V^* \otimes W^*$ is irreducible.
- (iii) $V^\kappa \otimes W^\kappa$ is irreducible.
- (iv) $V^- \otimes W^-$ is irreducible.

Proof. The equivalence of the four statements is an immediate consequence of Proposition 5.1.4 together with (5.1.9) and (5.1.8) and the fact that the Grothendieck ring of $\tilde{\mathcal{C}}_q$ is commutative. If V is an increasing minimal affinization and $e = n$, the first statement is part of Theorem 2.5.1. If V is increasing and $e = 1$, then V^κ is increasing, $\bar{e} = n$ and, hence, $V^\kappa \otimes W^\kappa$ has length at most 2 by Theorem 2.5.1. The case that V is decreasing and $e = 1$ follows by a similar argument using dualization instead of κ . Finally, if V is decreasing and $e = n$, we apply dualization and κ to obtain a tensor product as that of Theorem 2.5.1. \square

By Lemma 5.1.2, there exists a unique pair $(\tilde{\omega}, \tilde{\varpi}) \in \mathcal{P}^+ \times \mathcal{P}^+$ such that

$$(\omega, \varpi) = \begin{cases} (\tilde{\omega}^*, \tilde{\varpi}^*), & \text{if } V \text{ is decreasing and } e = 1; \\ (\tilde{\omega}^\kappa, \tilde{\varpi}^\kappa), & \text{if } V \text{ is increasing and } e = 1; \\ (\tilde{\omega}^-, \tilde{\varpi}^-), & \text{if } V \text{ is decreasing and } e = n. \end{cases}$$

Moreover, by Proposition 5.1.3, $V_q(\tilde{\omega})$ is an increasing minimal affinization and $V_q(\tilde{\varpi})$ is a Kirillov-Reshetikhin module with $\text{wt}(\tilde{\varpi}) = k\omega_n$. In other words, Theorem 2.5.1 applies to $V_q(\tilde{\omega}) \otimes V_q(\tilde{\varpi})$. In the case that $V \otimes W$ is reducible and, hence, so is $V_q(\tilde{\omega}) \otimes V_q(\tilde{\varpi})$ by the previous corollary, we denote by $\tilde{\lambda}$ the Drinfeld polynomial of the extra irreducible factor of $V_q(\tilde{\omega}) \otimes V_q(\tilde{\varpi})$ and by λ' that of the extra irreducible factor of $V \otimes W$. Set

$$i_1 = \min(\text{supp}(\lambda))$$

and note that

$$\bar{i}_1 = \max(\text{supp}(\lambda^*)).$$

We are ready to prove the completion of Theorem 2.5.1

Corollary 5.2.2.

(a) If V is decreasing and $e = 1$, then $V \otimes W$ is reducible if and only if there exist $p \in \text{supp}(\lambda)$ and $k' > 0$ such that either one of the following options hold:

- (i) $k' \leq \min\{\lambda(h_p), k\}$ and $r + 2k + p + 1 = r_p + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$ and $r_{i_1} + 2\lambda(h_{i_1}) + i_1 + 1 = r + 2k'$ and $p = \min\{i : |\lambda|_i \geq k'\}$.

In both cases, $\lambda' = \tilde{\lambda}^*$.

(b) If V is increasing and $e = 1$, then $V \otimes W$ is reducible if and only if there exist $p \in \text{supp}(\lambda)$ and $k' > 0$ such that either one of the following options hold:

- (i) $k' \leq \min\{\lambda(h_p), k\}$ and $r_p + 2\lambda(h_p) + p + 1 = r + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$ and $r + 2k + i_1 + 1 = r_{i_1} + 2k'$ and $p = \min\{i : |\lambda|_i \geq k'\}$.

In both cases, $\lambda' = \tilde{\lambda}^\kappa$.

(c) If V is decreasing and $e = n$, then $V \otimes W$ is reducible if and only if there exist $p \in \text{supp}(\lambda)$ and $k' > 0$ such that either one of the following options hold:

- (i) $k' \leq \min\{\lambda(h_p), k\}$ and $r_p + 2\lambda(h_p) + n - p + 2 = r + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$ and $r + 2k + n - i_0 + 2 = r_{i_0} + 2k'$ and $p = \max\{i \in I : |\lambda|_i \geq k'\}$.

In both cases, $\lambda' = \tilde{\lambda}^-$.

Proof. We know that $V \otimes W$ is reducible if and only if $V_q(\tilde{\omega}) \otimes V_q(\tilde{\varpi})$ is. Therefore, the conditions listed above are obtained from those of Theorem 4.1.2 by applying κ and/or by considering dual modules. Thus, assume $V_q(\tilde{\omega}) \otimes V_q(\tilde{\varpi})$ is reducible and consider the exact sequence

$$0 \rightarrow M \rightarrow V_q(\tilde{\omega}) \otimes V_q(\tilde{\varpi}) \rightarrow N \rightarrow 0.$$

We have either $M \cong V_q(\tilde{\lambda})$ or $N \cong V_q(\tilde{\lambda})$.

Assume first that we are in case (a). Thus, we also have the exact sequence

$$0 \rightarrow N^* \rightarrow W \otimes V \rightarrow M^* \rightarrow 0,$$

which immediately proves the statement about λ' . To obtain conditions (i) and (ii) above from those of Theorem 4.1.2, observe that, since $\tilde{\omega}^* = \omega$, (5.1.1) implies that

$$\tilde{\omega}_i = Y_{i, r_i^*, \lambda(h_i)} \quad \text{where} \quad r_i^* = r_i + h^\vee.$$

Notice also that $\tilde{\varpi} = Y_{n, r^*, k}$ with $r^* = r + h^\vee$ and that $|\lambda| = |\lambda^*|$. By Theorem 4.1.2, there exist a pair (p, k') such that $\bar{p} \in \text{supp}(\lambda^*)$, $k' > 0$, and either one of the following options hold:

- (i) $k' \leq \min\{\lambda^*(h_{\bar{p}}), k\}$ and $r^* + 2k + n - \bar{p} + 2 = r_p^* + 2k'$;

(ii) $k' \leq \min\{|\lambda|, k\}$ and $r_{i_1}^* + 2\lambda^*(h_{i_1}) + n - \bar{i}_1 + 2 = r^* + 2k'$, and $\bar{p} = \max\{i \in I : |\lambda^*| \geq k'\}$.

Recalling that $\bar{i} = n + 1 - i$, $\lambda^*(h_i) = \lambda(h_i)$, and using the expressions for r_i^* and r^* above, we get the conditions in the statement of part (a).

If we are in case (b), we have the exact sequence

$$0 \rightarrow M^\kappa \rightarrow W \otimes V \rightarrow N^\kappa \rightarrow 0,$$

which proves the statement about λ' as before. This time, it follows from (5.1.1) and (5.1.7) that

$$\tilde{\omega}_i = Y_{\bar{i}, r'_i, \lambda(h_i)} \quad \text{where} \quad r'_i = -r_i - 2(\lambda(h_i) - 1) + h^\vee$$

and

$$\tilde{\omega} = Y_{n, r', k} \quad \text{where} \quad r' = -r - 2(k - 1) + h^\vee.$$

By Theorem 4.1.2, there exist a pair (p, k') such that $\bar{p} \in \text{supp}(\lambda^*)$, $k' > 0$, and either one of the following options hold:

- (i) $k' \leq \min\{\lambda^*(h_{\bar{p}}), k\}$ and $r' + 2k + n - \bar{p} + 2 = r'_p + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$ and $r'_{i_1} + 2\lambda^*(h_{i_1}) + n - \bar{i}_1 + 2 = r' + 2k'$, and $\bar{p} = \max\{i \in I : |\lambda^*| \geq k'\}$.

Using the expressions for r'_i and r' above, we get the conditions in the statement of part (b).

Finally, in case (c), we have the exact sequence

$$0 \rightarrow (N^*)^\kappa \rightarrow V \otimes W \rightarrow (M^*)^\kappa \rightarrow 0,$$

which, together with (5.1.8), proves the statement about λ' . This time we have

$$\tilde{\omega}_i = Y_{i, r_i^-, \lambda(h_i)} \quad \text{where} \quad r_i^- = -r_i - 2(\lambda(h_i) - 1)$$

and

$$\tilde{\omega} = Y_{n, r^-, k} \quad \text{where} \quad r^- = -r - 2(k - 1).$$

By Theorem 4.1.2, there exist a pair $p \in \text{supp}(\lambda)$ and $k' > 0$ such that and either one of the following options hold:

- (i) $k' \leq \min\{\lambda(h_p), k\}$ and $r^- + 2k + n - p + 2 = r_p^- + 2k'$;
- (ii) $k' \leq \min\{|\lambda|, k\}$ and $r_{i_0}^- + 2\lambda(h_{i_0}) + n - i_0 + 2 = r^- + 2k'$, and $p = \max\{i \in I : |\lambda| \geq k'\}$.

Using the expressions for r_i^- and r^- above, we get the conditions in the statement of part (c). \square

5.3. Socle and Head. It remains to prove Corollary 2.5.2 and its analogues for the other cases given in Corollary 5.2.2. We begin by observing that, if $V, W \in \tilde{\mathcal{C}}_q$ are such that $V \otimes W$ is irreducible, then

$$(5.3.1) \quad V \otimes W \cong W \otimes V.$$

As we have seen in the previous subsections, (5.3.1) can be false if $V \otimes W$ is reducible.

It will be convenient to introduce the following notation. Given $V \in \tilde{\mathcal{C}}_q$ we will say that V has thin top if there exists $\lambda \in P^+$ satisfying:

- (1) $V_\lambda \neq 0$ and $\dim(V_\lambda) = 1$;
- (2) $V_\mu \neq 0$ only if $\mu \leq \lambda$.

Evidently, such λ is unique if it exists. Observe that every tensor product of modules with thin top has thin top as well. Given a module with thin top, let $\text{Top}(V)$ be the submodule generated by its top weight space. Our main extra tool for proving Corollary 2.5.2 is the following consequence of the main result of [4] (see also [21, Corollary 4.4]).

Proposition 5.3.1. Let $l \in \mathbb{Z}_{\geq 1}$, $i_j \in I$, $m_j \in \mathbb{Z}_{\geq 1}$, $a_j \in \mathbb{C}^\times$ for $j = 1, \dots, l$.

- (a) If $\frac{a_j}{a_k} \notin q^{\mathbb{Z}_{>0}}$ for $j > k$, then $V_q(\omega_{i_1, a_1, m_1}) \otimes \cdots \otimes V_q(\omega_{i_l, a_l, m_l})$ is a highest- ℓ -weight module.
 (b) If $\frac{a_j}{a_k} \notin q^{\mathbb{Z}_{<0}}$ for $j > k$, then $\text{Top}(V_q(\omega_{i_1, a_1, m_1}) \otimes \cdots \otimes V_q(\omega_{i_l, a_l, m_l}))$ is irreducible. \diamond

Henceforth, fix the notation of Theorem 2.5.1. In particular,

$$(5.3.2) \quad s_{i_0} > \cdots > s_2 > s_1,$$

where s_i defined in (2.5.1) and we have:

Corollary 5.3.2. $V_q(\omega_{i_0, a_{i_0}, \lambda(h_{i_0})}) \otimes \cdots \otimes V_q(\omega_{2, a_2, \lambda(h_2)}) \otimes V_q(\omega_{1, a_1, \lambda(h_1)})$ is highest- ℓ -weight and

$$V_q(\omega) \cong \text{Top}\left(V_q(\omega_{1, a_1, \lambda(h_1)}) \otimes V_q(\omega_{2, a_2, \lambda(h_2)}) \otimes \cdots \otimes V_q(\omega_{i_0, a_{i_0}, \lambda(h_{i_0})})\right).$$

\diamond

Suppose $V = V_q(\omega) \otimes V_q(\varpi)$ is reducible, and let (p, k') be the pair satisfying either condition (i) or (ii) of the theorem.

Corollary 5.3.3. $V_q(\omega_{i, a_i, \lambda(h_i)}) \otimes V_q(\varpi)$ is irreducible for every $i \in \text{supp}(\lambda) \setminus \{p\}$. In particular, $V_q(\omega_{i, a_i, \lambda(h_i)}) \otimes V_q(\varpi) \cong V_q(\varpi) \otimes V_q(\omega_{i, a_i, \lambda(h_i)})$.

Proof. If it were reducible, using Theorem 2.5.1 with $\omega_{i, a_i, \lambda(h_i)}$ in place of ω , it would follow that there would exist $0 < k'' \leq \min\{\lambda(h_i), k\}$ such that the pair (i, k'') satisfies either condition (i) or (ii) of Theorem 2.5.1. But then (i, k'') satisfies the same condition for ω as well contradicting the uniqueness of the pair (p, k') . The second statement is immediate from the first and Proposition 3.1.1. \square

We are ready to prove Corollary 2.5.2. To shorten notation, write

$$V_i = V_q(\omega_{i, a_i, \lambda(h_i)}), \quad i \leq i_0.$$

If neither conditions (i) nor (ii) are satisfied, then V and V' are irreducible and, hence, highest- ℓ -weight. Otherwise, suppose first that the pair (p, k') satisfies condition (i) and let us show that V is highest- ℓ -weight. Corollary 5.3.2 implies that we have surjective map

$$V_{i_0} \otimes \cdots \otimes V_2 \otimes V_1 \otimes V_q(\varpi) \twoheadrightarrow V.$$

Condition (i) implies that $s_p > s$ and, hence, there exists $i_1 \leq p, i_1 \in \text{supp}(\lambda)$, such that $s_{i_1} > s$ and $s \geq s_i$ for $i \leq i_1$. Corollary 5.3.3 then implies that we have a surjective map

$$(5.3.3) \quad V_{i_0} \otimes \cdots \otimes V_{i_1} \otimes V_q(\varpi) \otimes V_{i_1-1} \otimes \cdots \otimes V_2 \otimes V_1 \twoheadrightarrow V.$$

Since, by choice of i_1 , we have

$$(5.3.4) \quad s_{i_0} > \cdots > s_{i_1} > s \geq s_{i_1-1} > \cdots > s_2 > s_1,$$

Proposition 5.3.1 then implies that the left-hand side of (5.3.3) is highest- ℓ -weight. Hence, so is V . This proves the indecomposability of V as well as the first exact sequence of Corollary 2.5.2.

By inverting the order of all tensor products in the above argument and working with injective maps instead of surjective ones, one can similarly show that

$$V_q(\lambda) \cong \text{Top}(V') \subsetneq V'$$

which proves the second exact sequence.

In the case that condition (ii) holds, then $s > s_p$ and the argument is similar. We omit the details. In particular, it follows that V' is indecomposable. The following is now immediate from the proof of Corollary 5.2.2.

Corollary 5.3.4. Let V and W be as in Corollary 5.2.2.

- (a) If V is decreasing and $e = 1$, then $V \otimes W$ is not highest- ℓ -weigh if and only if (ii) holds while $W \otimes V$ is not highest- ℓ -weigh if and only if (i) holds.
- (b) If V is increasing and $e = 1$, then $V \otimes W$ is not highest- ℓ -weigh if and only if (i) holds while $W \otimes V$ is not highest- ℓ -weigh if and only if (ii) holds.
- (c) If V is decreasing and $e = n$, then $V \otimes W$ is not highest- ℓ -weigh if and only if (ii) holds while $W \otimes V$ is not highest- ℓ -weigh if and only if (i) holds. \diamond

To complete the proof of Corollary 2.5.2, it remains to show that V' is indecomposable if (i) holds and, similarly, that V is indecomposable if (ii) holds. We shall write down the proof of the latter. Let $\hat{\omega}$ and $\hat{\varpi}$ be such that

$$\hat{\omega}^* = \omega \quad \text{and} \quad \hat{\varpi}^* = \varpi.$$

By part (a) of Corollary 5.3.4, $\hat{V} = V_q(\hat{\omega}) \otimes V_q(\hat{\omega})$ is highest- ℓ -weight and, hence, indecomposable. But then, $V \cong \hat{V}^*$ must also be indecomposable.

Remark 5.3.5. Since W is a real simple object of $\tilde{\mathcal{C}}_q$ in the sense of [18], it follows from [20, Theorem 3.12] that V and V' have simple socle and head from where their indecomposability is easily deduced. The proof of [20, Theorem 3.12] is based on an analysis of the action of the R -matrix, which is not required in our approach. \diamond

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